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An operator $T \in \mathcal{L}(\ell_p(A), \ell_p(B))$ is extreme contraction if it is an extreme point of unit ball (A,B - index sets, $\ell_p(A)$ - - Banach space (over $\mathbb{R}$ or $\mathbb{C}$) of all p-summable functions on A). Let $1 \le p \le \infty$.

For $p=\infty$ we can characterize all extreme contractions as the lattice homomorphisms taking 1 into 1 multiplied by functions of absolute value 1 [M.Shaver, Israel J. Math. 12 (1972), C.Kim, Math. Zeitsch. 151 (1976), A.Iwanik, Colloq.Math. 40] .

For $p=1$ and real ℓ_1-space extr. contr. can be characterized (by duality) [Iwanik, Kim] .

For $p=2$ and field $\mathbb{C}$ the set of extr. contr. coincides with the set of all isometries and coisometries [Kadison, Ann. Math. 54 (1951)] .

For $\alpha \in A$ we denote by e_α the element of $\ell_p(A)$ defined by $e_\alpha(\gamma) = \delta_{\alpha\gamma}$, $\gamma \in A$. The index family $(e_\alpha)_{\alpha \in A}$ forms the canonical basis of E .

To every operator $T \in \mathcal{L}(\ell_p(A), \ell_p(B))$ there corresponds a unique matrix with scalar entries $(t_{\beta\alpha})$, $\alpha \in A$, $\beta \in B$ s.t. the α-th column represents Te_α in the canonical basis (e_β) of $\ell_p(B)$.

According to the behaviour of T , we will partition the index sets A,B into disjoint subsets A_i, B_i (i=0,1,2,3,4) . Let $A_0 = \{\alpha \in A ,\ t_{\beta\alpha} = 0 \text{ for all } \beta \in B\}$, $B_0 = \{ \beta \in B ,$

$t_{\beta\alpha} = 0$ for all $\alpha \in A \}$. Next let C be the set of all elements $\alpha \in A$ such that:

(1) there exists a $\beta \in B$ with $t_{\beta\alpha} \neq 0$ and

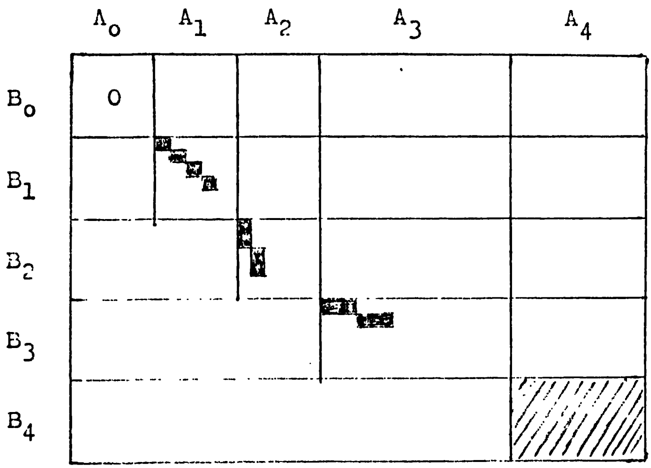
(2) if $t_{\beta\alpha} = 0$ for some $\beta \in B$, then $t_{\beta\gamma} = 0$ for all $\gamma \neq \alpha$.

Now we define A_1 to be the set of all elements $\alpha \in A$ s.t. $t_{\beta\alpha} \neq 0$ for only one $\beta \in B$ and we put $A_2 = C \setminus A_1$. Let A_3 be the set of all $\alpha \in A \setminus A_1$ such that:

(i) there exists exactly one $\beta \in B$ with $t_{\beta\alpha} \neq 0$ and

(ii) $t_{\beta\gamma} \neq 0 \Rightarrow t_{\delta\gamma} = 0$ for all $\delta \neq \beta$. Finally we put $A_4 = A \setminus (\bigcup_{i=0}^{3} A_i)$.

For $i=1,2,3,4$ let $B_i = \{ \beta \in B \, , \, t_{\beta\alpha} \neq 0$ for some $\alpha \in A_i \}$ (Fig. 1).



Theorem 1. Let $1 < p < \infty$, $p \neq 2$, $T \in \mathcal{L}(\ell_p(A), \ell_p(B))$ and let $A_4 = \emptyset$, $\|T\| \leq 1$. Then T is an extreme contraction iff the following two conditions are satisfied.

(a) $\|Te_\alpha\| = 1$ for $\alpha \in A$ and $\|Te_\beta\| = 1$ for $\beta \in B$,

(b) $A_o = \emptyset$ or $A_2 = B_o = \emptyset$ in the case of $1 < p < 2$

and

$B_o = \emptyset$ or $B_3 = A_o = \emptyset$ in the case of $2 < p < \infty$.

Corollary. For $p \neq 2$ $(1 < p < \infty)$ the set of all extreme contractions on the ℓ_p-space $(\dim \geq 2)$ is not closed.

Let X denote the two-dim ℓ_p-space.

Theorem 2. Let $1 < p < \infty$, $p \neq 2$ and $T \in \mathcal{L}(X,X)$, $\|T\| = 1$. Then T is an extreme contraction iff either T attains its norm in two linearly independent vectors in X or T is of the form

1^o $T = X \otimes e_i$ in the case of $1 < p < 2$

2^o $T = e_i \otimes y$ in the case of $2 < p < \infty$

with $x,y \neq e_j$ $(i,j=1,2)$, $\|x\| = \|y\| = 1$, i.e.

$$\underset{X^*}{\underset{\cap}{x}} \otimes \underset{X}{\underset{\cap}{y}} : X \longrightarrow X , \quad (x \otimes y)(z) = \langle z,x \rangle y , \quad z \in X .$$