On the existence problem in the algebraic approach to quantum field theory

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1. The algebraic approach to quantum field theory

In the framework of Garding-Wightman-Axioms /4/ a quantum field $\varphi(x)$ (neutral, scalar) is described by a normed, positive linear functional W on the algebra of test functions

$$\mathcal{S}_{\otimes} = \mathbb{C} \oplus \mathcal{S}(\mathbb{R}^{d}) \oplus \mathcal{S}(\mathbb{R}^{2d}) \oplus \dots \qquad /1,12/,$$

($\mathcal{S}(\mathbb{R}^{nd})$ is the Schwartz space over $\mathbb{R}^{nd}$ and d is the space-time dimension). W has additional properties listed below and reflecting the GW-axioms.

The elements of $\mathcal{S}_{\otimes}$ have the form $f=(f_0, f_1, \dots, f_n, 0, \dots)$ where only a finite number of components $f_i \in \mathcal{S}(\mathbb{R}^{id})$ is different from zero. $\mathcal{S}_{\otimes}$ becomes a $*$-algebra with unity $\mathbb{1}=(1,0,0,\dots)$. The operations are given by $(f+g)_m = f_m + g_m$, $(fg)_m = \sum_{i+j=m} f_i g_j$, $(f^*)_m = \bar{f}_m(x_m, \dots, x_1)$. Let $K = \{\sum_{i=1}^{N} f^{(i)*} f^{(i)};\ f^{(i)} \in \mathcal{S}_{\otimes},\ N=1,2,\dots\}$ be the cone of positive elements of $\mathcal{S}_{\otimes}$.

In a concentrated formulation a Wightman functional W is a linear functional on $\mathcal{S}_{\otimes}$ with

i) $W(\mathbb{1})=1$, ii) $W(f) \geq 0$ for all $f \in K$, iii) $W(f)=0$ for all $f \in L$,
iv) W is continuous, (A)

(L is a certain subspace of $\mathcal{S}_{\otimes}$ related to the Poincaré invariance, locality and spectrality).

(A) means geometrically the following:

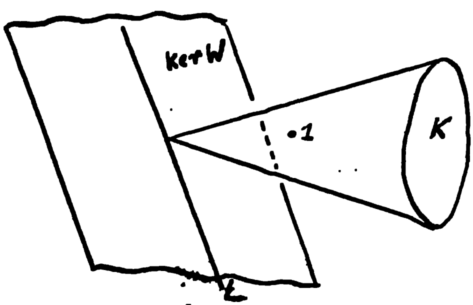

Figure 1

By the GNS-representation of $\mathcal{S}_\otimes$ there is a one-to-one correspondence between the quantum fields $\varphi(x)$ and the Wightman-functionals W /1,12/. Therefore, in some sense one can say that axiomatic quantum field theory is a mathematical problem, namely the investigation of $\mathcal{S}_\otimes$, its positive linear functionals, the study of K and L and so on.
The algebraic structure of $\mathcal{S}_\otimes$ is investigated in /2,13/.

Let $\mathfrak{S}_n$ denote the well-known Schwartz space topology on $\mathcal{S}(\mathbb{R}^{nd})$, for instance given by the following system of semi norms:

$$\|f_n\|_m = \sup_x \sup_{r_i^j \le m} \left| \prod_{i=1}^{n} \prod_{j=0}^{d-1} (1+(x_i^j)^2)^m \left(\frac{\partial}{\partial x_i^j}\right)^{r_i^j} f_n(x_1;\dots,x_n)\right|, \; m=0,1,\dots,$$

$(x_i=(x_i^0,x_i^1,\dots,x_i^{d-1}))$.

Then we can define a lot of topologies on $\mathcal{S}_\otimes$.

Definition:

i) $\tau_\otimes$: $p_{(\gamma_n)(\nu_n)}(f) = \sum_{n\ge 0} \gamma_n \|f_n\|_{\nu_n}$, where (γ_n) and (ν_n) run through the set of all sequences of natural numbers.

ii) τ_∞ : $p_{(\gamma_n)k}(f) = \sum_{n\ge 0} \gamma_n \|f_n\|_k$, where (γ_n) runs through the set of all sequences of natural numbers but k=0,1,... is fixed in every semi norm, /9/.

iii) τ_P : $q_{n,m}(f) = \|f_n\|_m$, n,m=0,1,2,... .

iv) $\mathcal{N}$: $\hat{p}(f) = \inf \left\{ \sum p(g^{(i)}) p(h^{(i)});\; f=\sum g^{(i)} h^{(i)} \right\}$ and p runs through the set of $\tau_\otimes$-continuous semi norms, /15/.

Let us remark that $\tau_\otimes$ is the topology of the direct sum and τ_P the restriction of the topology of the direct product $\bigtimes_{n=0}^{\infty} \mathcal{S}(\mathbb{R}^{nd})$ to the subspace $\mathcal{S}_\otimes$. Thus, $\tau_\otimes$ is the strongest l.c. topology on $\mathcal{S}_\otimes$ such that the restriction of $\tau_\otimes$ to every subspace $\mathcal{S}(\mathbb{R}^{nd})$ (n=1,2,...) is the Schwartz space topology $\mathfrak{S}_n$ while τ_P is the weakest topology with this property. Some properties of the l.c. topologies on $\mathcal{S}_\otimes$ between τ_P and $\tau_\otimes$ are listet in the following figure 2. Figure 2 shows that the topology $\tau_\otimes$ is a "good" one from topological viewpoint but a "bad" one from viewpoint of semiordering and τ_∞, $\mathcal{N}$ for instance are "good" ones from viewpoint of semiordering but "bad" ones from viewpoint of the topological structure.

Figure 2: /5/

Let τ be a l.c. topology on $\mathcal{S}_\otimes$	τ_β . . . τ_∞ . . . $\mathcal{N}$ . . . $\tau_\otimes$
The restriction of τ to $\mathcal{S}_\otimes$	$\tau \restriction \mathcal{S}(\mathbb{R}^{nd}) = \mathcal{S}_n$
The closure of a set $M \subset \mathcal{S}_\otimes$	$\overline{M}^{\tau_\beta} = \overline{M}^{\tau} = \overline{M}^{\tau_\otimes}$, if M fulfils $S_n M \subset M$, n=0,1,...
Is $\mathcal{S}_\otimes[\tau]$ complete?	Yes, if there is a +.+.+.+.+.+.+.+.+.+.+.+ filter base $\mathcal{U}[\tau]$ with $S_n U \subset U$, n=0,1,..., $U \in \mathcal{U}[\tau]$
The τ-bounded sets	+++++++++++++++++++++++ the same bounded sets
Is $\mathcal{S}_\otimes[\tau]$ barrelled?	..................................+
Is $\mathcal{S}_\otimes[\tau]$ bornological?	+.+.+.+.+.+.+ ....+
Is K τ-normal?	. + .+ .+ .+ .+ .+ .+ . + .+..........

(+ means "yes" but . means "no", S_n denotes the projection from $\mathcal{S}_\otimes$ onto $\mathbb{C} \oplus \mathcal{S}(\mathbb{R}^d) \oplus \mathcal{S}(\mathbb{R}^{2d}) \oplus \ldots \oplus \mathcal{S}(\mathbb{R}^{nd})$.)

Because of (Aii) it is of interest to investigate the cone K and the positive linear functionals. There is a linear functional T with $T(f) \geq 0$ for all $f \in K$ and T is not $\tau_\otimes$-continuous. But if one replaces K by its closure $\overline{K}^{\tau_\otimes}$ one can prove the following

Statement:

a) Every linear functional T on $\mathcal{S}_\otimes$ with $T(f) \geq 0$ for all $f \in \overline{K}^{\tau_\otimes}$ is automatically $\mathcal{N}$-continuous, /14/.

b) It is $\overline{K}^{\tau_\infty} = \overline{K}^{\tau_\otimes} = \left\{ \sum_{i=1}^{\infty} f^{(i)*} f^{(i)};\ f^{(i)} \in \mathcal{S}_\otimes \right.$, the sum is $\left. \tau_\infty\text{-convergent} \right\}$ /3,6/.

Because of a) it is possible to replace (Aii) and (Aiv) by the new condition (A'ii): $W(f) \geq 0$ for all $f \in \overline{K}^{\tau_\otimes}$.

A consequence of b) is that if one want to closure K it is sufficient to do this with sequences and not with nets. Some further interesting properties of K and $\overline{K}^{\tau_\otimes}$ are proved in /6,11,14/.

2. On the existence problem in axiomatic field theory

The axiomatic formulation of quantum field theory leads to the following three questions:

A) Are there contradictions between the axioms?

B) Are the GW-axioms independent?

C) Are the axioms general enough to describe interacting fields also?

The answer to A) is no, because there are the free fields for instance. The answer to B) is yes. But the answer to C) is almost unknown. One has examples of interacting fields for space-time-dimension 2 and 3 only. The following two theorems show the existence of fields which are different from the known fields. The proofs are abstract ones, i.e. we do not explicitely construct the fields.

Theorem 1: /7/

a) Every Wightman functional is $\mathcal{N}$-continuous.

b) The Wightman functionals of the free fields are τ_2-continuous. (τ_2 : $p_{(\gamma_n),2}(f)=\sum\gamma_n\|f_n\|_2$, (γ_n) runs through the set of all sequences of natural numbers.)

c) The Wightman functionals of the generalized free fields are τ_∞-continuous.

d) The Wightman functionals of the Wick polynomials and their derivatives are τ_∞-continuous.

e) The superposition of Wightman functionals of a)...d) is τ_∞-continuous too.

Theorem 2: /7/

There are Wightman functionals which are not τ_∞-continuous.

Theorem 1 shows that the Wightman functionals of all known fields are τ_∞-continuous. On the other hand Theorem 2 implies the existence of fields without this property. Further the topologies on $\mathcal{S}_\otimes$ give a possibility to classify the Wightman functionals by the help of their continuity with respect to these topologies.

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