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SEVENTH WINTER SCHOOL (1979)

On the existence problem in the algebraic
approach to quantum field theory

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1. The algebraic approach to quantum field theory

In the framework of Garding-Wightman-Axioms /4/ a quantum field $\varphi(x)$ (neutral, scalar) is described by a normed, positive linear functional W on the algebra of test functions

$$\mathcal{S}_0 = \mathbb{C} \oplus \mathcal{S}(\mathbb{R}^d) \oplus \mathcal{S}(\mathbb{R}^{2d}) \oplus \dots \quad /1,12/,$$

($\mathcal{S}(\mathbb{R}^{nd})$ is the Schwartz space over \mathbb{R}^{nd} and d is the space-time dimension). W has additional properties listed below and reflecting the GW-axioms.

The elements of \mathcal{S}_0 have the form $f = (f_0, f_1, \dots, f_n, 0, \dots)$ where only a finite number of components $f_i \in \mathcal{S}(\mathbb{R}^{id})$ is different from zero. \mathcal{S}_0 becomes a $*$ -algebra with unity $\mathbf{1} = (1, 0, 0, \dots)$. The operations are given by $(f+g)_m = f_m + g_m$, $(fg)_m = \sum_{i+j=m} f_i g_j$,

$(f^*)_m = \overline{f}_m(x_m, \dots, x_1)$. Let $K = \left\{ \sum_{i=1}^N f^{(i)} * f^{(i)}; f^{(i)} \in \mathcal{S}_0, N=1,2,\dots \right\}$

be the cone of positive elements of \mathcal{S}_0 .

In a concentrated formulation a Wightman functional W is a linear functional on \mathcal{S}_0 with

- i) $W(\mathbf{1})=1$, ii) $W(f) \geq 0$ for all $f \in K$, iii) $W(f)=0$ for all $f \in L$, (A)

iv) W is continuous, (L is a certain subspace of \mathcal{S}_0 related to the Poincaré invariance, locality and spectrality).

(A) means geometrically the following:

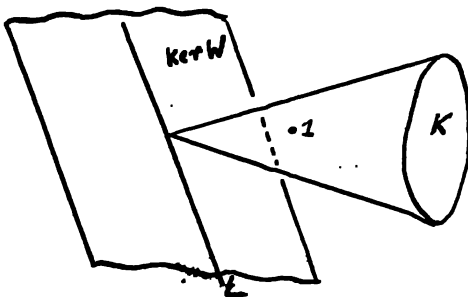


Figure 1

By the GNS-representation of \mathcal{F}_0 there is a one-to-one correspondence between the quantum fields $\mathcal{P}(x)$ and the Wightman-functionals $W /1,12/$. Therefore, in some sense one can say that axiomatic quantum field theory is a mathematical problem, namely the investigation of \mathcal{F}_0 , its positive linear functionals, the study of K and L and so on.

The algebraic structure of \mathcal{F}_0 is investigated in /2,13/.

Let \mathcal{S}_n denote the well-known Schwartz space topology on $\mathcal{S}(\mathbb{R}^{nd})$, for instance given by the following system of semi norms:

$$\|f_n\|_m = \sup_x \sup_{\substack{n \\ r_i \leq m}} \left| \prod_{i=1}^n \prod_{j=0}^{d-1} (1+(x_i^j)^2)^m \left(\frac{\partial}{\partial x_i} \right)^{r_i} f_n(x_1, \dots, x_n) \right|, \quad m=0,1,\dots,$$

$$(x_i = (x_i^0, x_i^1, \dots, x_i^{d-1})).$$

Then we can define a lot of topologies on \mathcal{F}_0 .

Definition:

- i) τ_0 : $p_{(\mathbf{r}_n)(\mathbf{v}_n)}(f) = \sum_{n \neq 0} \mathbf{r}_n \|f_n\|_{\mathbf{v}_n}$, where (\mathbf{r}_n) and (\mathbf{v}_n) run through the set of all sequences of natural numbers.
- ii) τ_∞ : $p_{(\mathbf{r}_n)k}(f) = \sum_{n \neq 0} \mathbf{r}_n \|f_n\|_k$, where (\mathbf{r}_n) runs through the set of all sequences of natural numbers but $k=0,1,\dots$ is fixed in every semi norm, /9/.
- iii) τ_p : $q_{n,m}(f) = \|f_n\|_m$, $n, m=0,1,2,\dots$.
- iv) \mathcal{N} : $\hat{p}(f) = \inf \left\{ \sum p(g^{(i)}) p(h^{(j)}); f = \sum g^{(i)} h^{(j)} \right\}$ and p runs through the set of τ_0 -continuous semi norms, /15/.

Let us remark that τ_0 is the topology of the direct sum and τ_p the restriction of the topology of the direct product $\prod \mathcal{S}(\mathbb{R}^{nd})$ to the subspace \mathcal{F}_0 . Thus, τ_0 is the strongest l.c. topology on \mathcal{F}_0 such that the restriction of τ_0 to every subspace $\mathcal{S}(\mathbb{R}^{nd})$ ($n=1,2,\dots$) is the Schwartz space topology \mathcal{S}_n while τ_p is the weakest topology with this property. Some properties of the l.c. topologies on \mathcal{F}_0 between τ_p and τ_0 are listed in the following: figure 2. Figure 2 shows that the topology τ_0 is a "good" one from topological viewpoint but a "bad" one from viewpoint of semiordering and τ_∞, \mathcal{N} for instance are "good" ones from viewpoint of semiordering but "bad" ones from viewpoint of the topological structure.

- B) Are the GW-axioms independent?
 C) Are the axioms general enough to describe interacting fields also?

The answer to A) is no, because there are the free fields for instance. The answer to B) is yes. But the answer to C) is almost unknown. One has examples of interacting fields for space-time-dimension 2 and 3 only. The following two theorems show the existence of fields which are different from the known fields. The proofs are abstract ones, i.e. we do not explicitly construct the fields.

Theorem 1: /7/

- a) Every Wightman functional is \mathcal{N} -continuous.
- b) The Wightman functionals of the free fields are τ_2 -continuous. ($\tau_2 : \mathcal{F}_{(\gamma_n), 2}(f) = \sum \gamma_n \|f_n\|_2$, (γ_n) runs through the set of all sequences of natural numbers.)
- c) The Wightman functionals of the generalized free fields are τ_∞ -continuous.
- d) The Wightman functionals of the Wick polynomials and their derivatives are τ_∞ -continuous.
- e) The superposition of Wightman functionals of a)...d) is τ_∞ -continuous too.

Theorem 2: /7/

There are Wightman functionals which are not τ_∞ -continuous.

Theorem 1 shows that the Wightman functionals of all known fields are τ_∞ -continuous. On the other hand Theorem 2 implies the existence of fields without this property. Further the topologies on \mathcal{S}_0 give a possibility to classify the Wightman functionals by the help of their continuity with respect to these topologies.

References:

- /1/ Borchers, H.J.: On the structure of the algebra of field operators. Nuovo Cimento, 24, 214 (1962).
- /2/ - : Algebraic aspects of Wightman field theory. In: Sen & Weil (Eds.), Statistical mechanics and field theory. New York 1972.
- /3/ - : Prepublication de la RCP n° 25, Vol.15, Strasbourg, 1973.
- /4/ Garding, L.; Wightman, A.S.: Fields as operator valued distributions in relativistic quantum field theory. Arkiv f. Fysik, 28, 129 (1964).
- /5/ Hofmann, G.: Topologies on the algebra of test functions. Dubna-preprint, JINR E2-10763 (1977).

- /6/ - : The closure of cones in the algebra of test functions. Rep.Math.Phys. 13, 187 (1978).
- /7/ - : Rigorous existence proofs for quantum fields, I. Preprint KMU-MPh-3 (1978). On the existence of quantum fields in space-time-dimension 4. to appear in Rep.Math.Phys.
- /8/ - ; Laßner,G.: Existence of Wightman type functionals. Wiss. Zeitschr. d.KMU, Leipzig, Math.-Naturw.R. 24, 535 (1975).
- ✓9/ Laßner,G.: On the algebra of test functions. Dubna-preprint JINR E2-5254 (1970).
- /10/ - : O^{∞} -topologies on the algebra of test functions. Publ.Dép. Math. Lyon 12/1, 25 (1975).
- /11/ - ; Uhlmann,A.: On positive functionals on algebras of test functions. Comm.math.Phys. 7, 152 (1968).
- /12/ Uhlmann,A.: Über die Definition der Quantenfelder nach Wightman und Haag. Wiss.Zeitschr.d.KMU, 11, 213 (1962).
- /13/ Wyss,W.: The field algebra and its positive linear functionals. Comm.math.Phys. 26, 223 (1972).
- /14/ - : in Lectures in theoretical physics, Vol. XI.D,Gordon & Breach Intersciences Publishers, Boulder 1968.
- /15/ Yngvason,J.: On the algebra of test functions for field operators. Comm.math.Phys. 27, 223 (1972).