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Invalid Vitali theorems


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Vitali type covering theorems in finite dimensional Banach spaces hold (under some regularity assumptions on the considered covers) for arbitrary measures (see [4]). If we drop the assumption of finite dimensionality the situation becomes different. By a result of Davies [3] there exist distinct probability measures on a metric space which agree on all balls. Although this particular behaviour is not possible in the case of Hilbert spaces, it was shown in [5] that Vitali Theorem does not hold for centered balls and Gaussian measures. The following result shows that even the Density Theorem does not hold in infinitely dimensional Hilbert spaces.

**Theorem.** Let $H$ be a separable infinitely dimensional real Hilbert space. Then there is a finite measure $u$ on the Borel $\sigma$-algebra of $H$ and a compact set $C \subseteq H$ such that $u(C) > 0$ and \[
\lim_{x \to x} \frac{u(C \cap B(x,r))}{u(B(x,r))} = 0 \quad \text{for each } x \in C.
\]

**Proof.** By induction one easily defines a sequence $\{a_k\}$ of positive numbers and a sequence $\{N_k\}$ of natural numbers such that $\sum_{k=1}^{\infty} a_k N_1 \ldots N_k < \infty$ and $\lim_{k \to \infty} a_k N_1 \ldots N_{k+1} = \infty$.

Let $S$ be the set of all finite sequences $(z_1, \ldots, z_k)$ of natural numbers such that $z_1 \leq N_1$ and let $Z$ be the set of all infinite sequences $(z_1, \ldots)$ of natural numbers such that $z_1 \leq N_1$.

For each $z = (z_1, \ldots, z_k) \in S$ choose $h(z) \in H$ such that
$\|n(z)\| = 2^{-k}$ and $h(y), h(z)$ are orthogonal whenever $y, z \in S, y \neq z$.

Put

$$g(z) = \sum_{j=1}^{k} h(z_1, \ldots, z_j) \quad \text{for } z = (z_1, \ldots, z_k) \in S.$$ 

$$f(z) = \sum_{j=1}^{3} h(z_1, \ldots, z_j) \quad \text{for } z = (z_1, \ldots) \in Z.$$ 

Note that $\|f(y) - f(z)\|^2 = 2^{-k+2}$ if $y, z \in Z, y \neq z$ and $k$ is the least natural number such that $z_k \neq y_k$ and

$\|f(z) - g(z_1, \ldots, z_k)\|^2 = 2^{-k}$ for each $z \in Z$ and natural $k$.

The set $Z$ considered as a product of finite topological spaces is a compact metrizable space. Let $v$ be the product of measures $v_j$ on the sets $\{1, \ldots, N_j\}$, where $v_j(n) = (N_j)^{-1}$.

Put $u = f(v) + \sum_{(z_1, \ldots, z_k) \in S} a_k \delta_{g(z_1, \ldots, z_k)}$, where $f(v)$ is the image measure and $\delta_x$ is the Dirac measure at $x$.

If $C = f(Z), z \in Z, x = f(z)$ and $2^{-k} \leq r^2 < 2^{-k+1}$ then $u(B(x, r) \cap C) = v\{y \in Z; y_i = z_i \text{ for } i = 1, \ldots, k+1\} = (N_1 \ldots N_{k+1})^{-1}$ and $u(B(x, r)) \geq a_k$, since $g(z_1, \ldots, z_k) \in B(x, r)$. Thus

$u(B(x, r) \cap C) \leq \frac{a_k (N_1 \ldots N_{k+1})^{-1}}{u(B(x, r))}.$

**Remark.** If we construct the sequences $\{a_k\} \downarrow \{\alpha_k\}$ so that

$\sum_{k=1}^{\infty} a_k N_1 \ldots N_k < 1$, then the measure $w = u - 2f(v)$ has the following properties

(i) $w(H) < 0$

(ii) for each $x \in H$ there is $r(x) > 0$ such that $w(B(x, r)) \geq 0$ for each positive $r < r(x)$.

This example should be compared with a recent result of Christensen [^]: If $u$ is a measure on $H$ such that for each $x \in H$
there exists $r(x) > 0$ such that $u$ vanishes on all balls contained in the ball with center $x$ and radius $r(x)$, then $u$ vanishes identically.

References.


