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WEAK COMPACTNESS IMPLIES STRONG COMPACTNESS IN THE SPACE OF UNIFORM MEASURES.

BY WALTER SCHACHERMAYER

Recall the definition [2]: A Saks-space is a triple $(E, \|\cdot\|, \tau)$, where $(E, \|\cdot\|)$ is normed space and τ is a locally convex topology on E such that OE, the $\|\cdot\|$- unitball of E is τ - closed anc τ-bounded. One defines the "mixed topology" γ to be the finest locally convex topology on E agreeing on OE with τ.

One may consider the following dual object $\{E'_\gamma, \|\cdot\|, (\mathcal{H}, \sigma(E'_\gamma, E$ where $(E'_\gamma, \|\cdot\|)$ is the Banach space of γ-continuous linear forms on E equipped with the dual norm of $(E, \|\cdot\|)$ and $\mathcal{H}$ is the family of γ-equicontinuous subsets of E'_γ equipped with the $\sigma(E'_\gamma, E)$ - topology for which the members of $\mathcal{H}$ are relatively compact. If $(E, \|\cdot\|, \tau)$ is a complete Saks-space, i.e. E is γ-complete, then by Grothendiecks completeness theorem one may recover E as the linear functionals on E'_γ such that the restriction to every H in $\mathcal{H}$ is $\sigma(E'_\gamma, E)$ -continuous ([5], th. IV. 6.2).

A typical example of such a dual object of a Saks-space is the following: Let X be a complete uniform space and define $(\mathcal{U}^b(X), \|\cdot\|_\infty)$ to be the Banach-space of uniformly continuous bounded real-valued functions on X equipped with the sup-norm. Let $\mathcal{H}$ be the family of uniformly equicontinuous bounded subsets of $\mathcal{U}^b(X)$ (abbreviated U.E.B.) equipped with the topology τ_p of pointwise convergence on X. The U.E.B.-sets are relatively compact in $\mathcal{U}^b(X)$ with respect to τ_p.

Define the space of "uniform measures on X" to be the space of linear functionals on $\mathcal{U}^b(X)$ such that the restriction to each U.E.B.-set is τ_p - continuous. If we equip $\mathcal{M}_u(X)$ with the norm dual to $\|\cdot\|_\infty$ and the topology γ of uniform convergence on the U.E.B.-sets it becomes a complete Saks-space.

Theorem: For a subset K in $\mathcal{M}_u(X)$ the following are equivalent:

(i) K is relatively $\sigma(M_u(X), \mathcal{U}^b(X))$ -compact

(ii) K is relatively γ - compact.

This theorem is due to Pachl [4], who proved it using rather delicate arguments. In [3] uniform measures were studied by using systematically the framework of Saks-spaces and Co-Saks-spaces and an easy proof of the above theorem was given there. I give here an outline of this proof; for details the reader is referred to [3].

Let's first give some motivating examples.

EXAMPLE I: If X is a uniformly discrete space, $\mathcal{U}^b(X)$ equals $l^\infty(X)$ and the U.E.B. - sets are just the bounded subsets of $l^\infty(X)$ equipped with the topology of pointwise convergence on X. $\mathcal{M}_u(X)$ is then $l^1(X)$ and γ is the topology of norm-convergence in $l^1(X)$. So the theorem reduces in this case to Schur's lemma that a weakly compact subset of $l^1(X)$ is norm-compact.

EXAMPLE II: If X is a compact Hausdorff-space, $\mathcal{U}^b(X)$ equals C(X), the space of continuous functions on X, and Ascoli's theorem implies that the U.E.B.-sets in $\mathcal{U}^b(X)$ are the relatively norm compact sets. $\mathcal{M}_u(X)$ is then the space of Radon-measures on X and γ is the topology of uniform convergence on compact subsets of C(X) So the theorem reduces in this case to the Banach-Dieudonné-theorem ([5], th. IV. 6.3.).

Let us now turn to the proof of the theorem.

It is shown in [3] that one may reduce to the case where X is a complete metric space by the use of some easy formal manipulations with projective and injective limits (taken in the proper categories!) So let us assume from now on that (X,d) is a complete metric space. As was shown in [1] the uniform measures then are exactly the bounded Radon measures on (X,d), i.e. the members μ of $(\mathcal{U}^b(X), \|\cdot\|_\infty)'$ that satisfy the following tightness condition.

(•) $\forall\varepsilon > 0$ there is a compact K in X s.t. for $f \in \mathcal{U}^b(X)$ $\|f\|_\infty \leq 1$ and f vanishing on K

$$|\langle f,\mu\rangle| < \varepsilon.$$

It was pointed out in [3] that this is also equivalent to the "Lipschitz-tightness" of μ, i.e. to the condition

(••) $\forall\varepsilon > 0$ there is a compact K in X s.t. for $f \in \mathcal{U}^b(X)$, $\| f \| \leq 1$, f vanishing on K and f obeying a Lipschitz - constant 1

$$|\langle f,\mu\rangle| < \varepsilon.$$

This notion gives rise to the crucial

<u>DEFINITION:</u> A subset K of $M_u(X)$ is called "uniformly Lipschitz tight" if for $\varepsilon > 0$ there is a compact K in X such that for $f \in \mathcal{U}^b(X)$ $\|f\|_\infty \leq 1$, f vanishing on K and f obeying a Lipschitz-constant 1

$$|\langle f,\mu\rangle| < \varepsilon. \qquad \text{for all } \mu \in K.$$

We shall show that in the case of a complete metric space (X,d) (i) and (ii) in the theorem are equivalent to

(iii) K is bounded and uniformly Lipschitz-tight.

Proof of the theorem: (ii) $\Rightarrow$ (i) : trivial.

(iii) $\Rightarrow$ (ii) : This is a consequence of the following elementary lemma [3] and the observation that on U.E.B.-sets the topologies of pointwise and compact convergence on X agree.

Lemma: Let H be a U.E.B.-set in $\mathcal{U}^b(X)$. Then for $\varepsilon > 0$ there is a constant M such that for every $g \in H$ there is $f \in \mathcal{U}^b(X)$, $\|f\|_\infty \leq M$ and f obeying a Lipschitz constant M, such that

$$\|f - g\|_\infty < \varepsilon.$$

(i) $\Rightarrow$ (iii): This is the essential part of the proof and we reduce the problem to Schur's lemma.
If K is relatively $\sigma(M_u, \mathcal{U}^b)$ - compact, then K is bounded in norm (say by 1) by the uniform boundedness theorem. So assume K is not uniformly Lipschitz - tight. We shall show how to construct an $\eta > 0$, a sequence $\{\mu_n\}_{n=1}^\infty$ in K and a sequence $\{f_n\}_{n=1}^\infty$ in $\mathcal{U}^b(X)$, $\|f_n\|_\infty \leq 1$, with pairwise disjoint supports in X and obeying a Lipschitz-constant η^{-1} such that

$$|\langle f_n, \mu_n \rangle| \geq \eta . \qquad n \in \mathbb{N}$$

Once this is done, we complete the proof as follows:
For any sequence $\{\lambda_n\}_{n=1}^\infty$ in l^∞, $\Sigma\lambda_n f_n$ (the sum taken pointwise) is in $\mathcal{U}^b(X)$ and obeys a Lipschitz - constant $\eta^{-1} . \|\{\lambda_n\}\|_\infty$.
Hence

$$T:\{\lambda_n\}_{n=1}^\infty \longrightarrow \Sigma\lambda_n f_n$$

is an operator from l^∞ to $\mathcal{U}^b(X)$ which sends bounded sets to U.E.B. sets. The transposed operator T' sends $M_u(X)$ into l^1 and T'(K) is a relatively $\sigma(l^1, l^\infty)$ compact set. This contradicts Schur's Lemma, as the n-th coordinate of $T'(\mu_n)$ is in absolute value greater or equal than η and $T'(K)$ therefore is not relatively norm-compact in l^1.

So let us carry out inductively the construction of $\{\mu_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$. By assumption there is $0 < \eta \leq 1$ so that for every compact set C in X there is a function f_C with $\|f_C\|_\infty \leq 1$, obeying a Lipschitz-constant 1 and vanishing off C and there is μ_C in K such that $|\langle f_C, \mu_C \rangle| \geq 4\eta$.

Define $\tilde{f}_C$ by

$$\tilde{f}_C : x \longrightarrow \begin{cases} 0 \text{ if } |f_C(x)| \leq 2\eta \\ f_C - 2\eta \text{ if } f_C(x) \geq 2\eta \\ f_C + 2\eta \text{ if } f_C(x) \leq -2\eta. \end{cases}$$

Then $\tilde{f}_C$ again is a function with $\|\tilde{f}_C\|_\infty \leq 1$ and obeying a Lipschitz-constant 1 but $\tilde{f}_C$ even vanishes on each point of x with distance from C less or equal than 2η . Still we have

$$|\langle \tilde{f}_C, \mu_C \rangle| \geq 2\eta.$$

We can now proceed with the construction. First find g_1 in $\mathcal{U}^b(X)$ and μ_1 in K with $\|g_1\|_\infty \leq 1$ and obeying a Lipschitz-constant 1, and such that $|\langle g_1, \mu_1 \rangle| \geq 2\eta$.

Since μ_1 is a Radon - measure on X we can find a compact set C_1 so that $|\mu_1| (X \setminus C_1) \leq \eta$.

Let $\qquad h_1 : x \longrightarrow [1 - \eta^{-1} d(x,C_1)]_+$.

Then $\qquad f_1 : x \longrightarrow [g_1(x) \wedge h_1(x)] \vee [-h_1(x)]$

is a function with $\|f_1\|_\infty \leq 1$ and obeying a Lipschitz - constant η^{-1} and vanishing for all x with distance from C_1 greater than η.

We have

$$|\langle f_1, \mu_1 \rangle| \geq \eta.$$

At the second step find g_2 in $\mathcal{U}^b(x)$ with $\|g_2\|_\infty \leq 1$, obeying a Lipschitz - constant 1, and vanishing for all x with $d(x,C_1) \leq 2\eta$ and find μ_2 in K with $|\langle g_2, \mu_2 \rangle| \geq 2\eta$. Since μ_2 is a Radon-measure on X we can find a compact subset C_2 of $\operatorname{supp}(g_2)$

the support of g_2, such that $|\mu_2|(\text{supp}(g_2) \setminus C_2) \leq \eta$.

Let $h_2 : x \longrightarrow [1 - \eta^{-1} d(x,C_1)]$.

Then $f_2 : x \longrightarrow [g_2(x) \wedge h_2(x)] \vee [-h_2(x)]$ is

a function with $\|f_2\|_\infty \leq 1$ obeying a Lipschitz - constant η^{-1} and vanishing for all x with distance from C_2 greater than η (whence in particular on the support of f_1). Further we have

$$|\langle f_2,\mu_2 \rangle| \geq \eta.$$

Continue in the same fashion to finish the induction, thus completing the proof of the theorem. q.e.d.

With exactly the same reduction to Schur's lemma one also proves the following

Corollary: Let $\{\mu_n\}_{n=1}^\infty$ be a weak Cauchy - sequence in $M_u(X)$. Then $\{\mu_n\}_{n=1}^\infty$ is γ - Cauchy and so γ - convergent.

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