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ON THE EXISTENCE OF G-COMPACTIFICATIONS

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A topological transformation group (ttg) is a triple $\langle G, X, \pi \rangle$ where G is a topological group, X is a topological space and π is an action of G on X , that is, $\pi: G \times X \rightarrow X$ is a continuous mapping such that

- (i) $\forall x \in X: \pi(e, x) = x$ (e denotes the identity element of G),
- (ii) $\forall (s, t, x) \in G \times G \times X: \pi(t, \pi(s, x)) = \pi(ts, x)$.

A ttg $\langle G, X, \pi \rangle$ will also be called a G -space. Using the notation $\pi^t x := \pi(t, x)$ for $(t, x) \in G \times X$, we have $\pi^e = 1_X$ and $\pi^t \circ \pi^s = \pi^{ts}$. So $t \mapsto \pi^t$ defines a homomorphism of G into the group of all autohomeomorphisms of X .

If $\langle G, X, \pi \rangle$ is a G -space and if there exists a G -space $\langle G, Y, \sigma \rangle$ such that Y is a compact Hausdorff space and X is (homeomorphic with) a dense subset of Y such that $\sigma^t|_X = \pi^t$ for every $t \in G$, then $\langle G, Y, \sigma \rangle$ is called a G -compactification of $\langle G, X, \pi \rangle$. A necessary condition for $\langle G, X, \pi \rangle$ to have a G -compactification is, that X is a Tychonoff space. The question is, whether of this condition is sufficient. For \mathbb{R} -spaces, this question occurs in [2], and for general G -spaces in [3] in the context of reflection of G -spaces in the category of compact Hausdorff G -spaces. The following result, which has just been published [4], provides a partial solution to this problem:

THEOREM 1. *If G is locally compact then every G -space $\langle G, X, \pi \rangle$ with X a Tychonoff space has a G -compactification.*

Another partial solution is included in Theorem 3 below, which is joint work of H. LUDEŠCHER (Timisoara, Romania) and myself. It is a consequence of the following theorem, in which the following notation will be used: $\pi_x^t := \pi(t, x) = \pi^t x$ for $(t, x) \in G \times X$; so $\pi_x: G \rightarrow X$ is continuous for all $x \in X$.

THEOREM 2. (BROOK, [1]). *Let $\langle G, X, \pi \rangle$ be a G -space in which X is a Tychonoff space, and suppose that there exists a uniformity \mathcal{U} for X (i.e. compatible with the topology of X) such that the following conditions are satisfied:*

- (i) $\{\pi_x: x \in X\}$ is \mathcal{U} -equicontinuous at e ;
- (ii) $\forall t \in G: \pi^t: X \rightarrow X$ is \mathcal{U} -uniformly continuous.

Then $\langle G, X, \pi \rangle$ has a G -compactification.

Proof (outline). Let \mathcal{W}^* be the weakest uniformity on X such that every \mathcal{U} -uniformly continuous function from X to the interval $[0, 1]$ is \mathcal{W}^* -uniformly

continuous. Then \mathcal{W}^* is compatible with the topology of X , and (X, \mathcal{W}^*) is pre-compact, i.e. the completion X^* of X w.r.t. \mathcal{W}^* is a compact Hausdorff space. Using (ii), it is easily seen that each $\pi^t: X \rightarrow X$ is \mathcal{W}^* -uniformly continuous, hence has a (uniformly) continuous extension $\sigma^t: X^* \rightarrow X^*$. Using condition (i), it is not difficult to show that the mapping $\sigma: (t, z) \mapsto \sigma^t z: G \times X^* \rightarrow X^*$ is continuous. Then $\langle G, X^*, \sigma \rangle$ is a G -compactification of $\langle G, X, \pi \rangle$. \square

THEOREM 3. Let $\langle G, X, \pi \rangle$ be a G -space with X a Tychonoff space, and suppose that there exists a uniformity \mathcal{U} for X such that $\{\pi^t: t \in G\}$ is \mathcal{U} -equicontinuous at every point of X . Then $\langle G, X, \pi \rangle$ has a G -compactification.

Proof (outline). For $\alpha \in \mathcal{U}$, define

$$G(\alpha) := \{(x, y) \in X \times X: (\pi_x^t, \pi_y^t) \in \alpha \text{ for all } t \in G\},$$

and let \mathcal{V} be the uniformity, generated by $\{G(\alpha): \alpha \in \mathcal{U}\}$. Then by \mathcal{U} -equicontinuity, \mathcal{V} is compatible with the topology of X . Moreover, $\{\pi^t: t \in G\}$ is \mathcal{V} -uniformly equicontinuous on X . Using this property of \mathcal{V} , it turns out that the collection of all sets of the form

$$[V, \alpha] := \{(\pi_x^t, \pi_y^s): s, t \in G \text{ \& } ts^{-1} \in V \text{ \& } (x, y) \in \alpha\},$$

V a neighbourhood of e in G and $\alpha \in \mathcal{V}$, is a base of a uniformity \mathcal{W} . Then \mathcal{W} turns out to be compatible with the topology of X as well. The following properties of \mathcal{W} and π are now easily established:

- (i) $\{\pi_x^t: x \in X\}$ is \mathcal{W} -equicontinuous at e
(indeed, for every neighbourhood V of e in G and every $\alpha \in \mathcal{V}$ we have $(\pi_x^t, x) = (\pi_x^t, x) \in [V, \alpha]$ for all $x \in X$, provided $t \in V$);
- (ii) $\forall t \in G: \pi^t: X \rightarrow X$ is \mathcal{W} -uniformly continuous
(indeed, if V is a neighbourhood of e in G and $\alpha \in \mathcal{V}$, then for every $t \in G$ there is a neighbourhood W of e in G such that $tWt^{-1} \subseteq V$, hence for all $(x, y) \in [W, \alpha]$ we have $(\pi_x^t, \pi_y^t) \in [V, \alpha]$). Now Theorem 2 implies that $\langle G, X, \pi \rangle$ has a G -compactification.

REMARK. In [3; 7.3.12] a different proof of Theorem 2 has been given. In fact, there we proved that condition (i) in Theorem 2 is sufficient for $\langle G, X, \pi \rangle$ to have a G -compactification $\langle G, Y, \sigma \rangle$ such that $w(Y) \leq \max\{w(G), w(X)\}$; here $w(Z)$ denotes the (topological) weight of a space Z .

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