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A NOTE ON MEASURABILITY OF TRAJECTORIES OF A STOCHASTIC PROCESS

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In this note, we present an alternative method of studying the measurability properties of trajectories of a stochastic process. More precisely, we will consider the following situation:

Given any vector or scalar measure

$$\vec{\mu}: \mathcal{B}(X^{<0,1>}) \rightarrow E$$

on a family $\mathcal{B}(X^{<0,1>})$ of all Baire subsets of $X^{<0,1>}$, where X is some locally compact metrizable space X , consider the question: On which trajectories lives $\vec{\mu}$?

Our method is based on the identification of each Baire function $f \in X^{<0,1>}$ (or, rather, the corresponding a.e. equivalence class) with the probability α_f on $<0,1> \times X$, determined uniquely by the requirements:

- i) α_f is carried by a graph of f
- ii) the projection $\pi(\alpha_f)$ of α_f on $<0,1>$ is just the Lebesgue measure.

Thus, by identifying f with α_f , the topology of convergence in measure on functions can be induced by the weak* topology on the space $\mathcal{P}(<0,1> \times \check{X})$ of all Radon probabilities on $<0,1> \times \check{X}$ (where $\check{X}$ denotes a one point compactification of X).

Now we generalize the notion of a trajectory.

Denote by

$$\mathcal{T} = \{ \alpha \in \mathcal{P}, \pi_{<0,1>}(\alpha) = \text{Lebesgue measure} \} \text{ "generalized trajectories"}$$

$\mathcal{T}_{\text{meas}} = \{ \alpha \in \mathcal{T}, \alpha \text{ is carried by a graph of some Baire function } f \in X^{<0,1>} \}$ "measurable trajectories"

$\mathcal{T}_{\text{cont}} = \{ \alpha \in \mathcal{T}, \alpha \text{ is carried by a graph of some a.e. continuous function } f \in X^{<0,1>} \}$ "a.e. continuous trajectories".

Now, we "carry on" the measure $\vec{\mu}: \mathcal{B}(X^{<0,1>}) \rightarrow E$ in some very natural way to the space $\mathcal{T}$.

Then we will investigate the support of the resulting measure.

We will show, at the end of this note, that each measure on

$\mathcal{T}_{\text{meas}}$ can be viewed as a measure on $X^{<0,1>}$.

Our results will then be comparable to the classical ones (see e.g. [1, Th.III,3,1]). Actually, they generalize them slightly.

Definition

Denote by $P_{IA}(\alpha) = \alpha(I \times A)$

($P_{IA}(\alpha)$ says, "how often α dwells in A during I ").

Suppose that each map

$$\{t_1 \dots t_n \rightarrow \vec{\mu}(\pi_{t_1 \dots t_n}^{-1}(A_1 \times \dots \times A_n))\} : <0,1>^n \rightarrow E$$

is Lusin measurable, whenever A_i are open in X .

$$\text{Put } \vec{\mu}(P_{I_1 A_1} \dots P_{I_n A_n}) = \int_{I_1 \times \dots \times I_n} \vec{\mu}(\pi_{t_1 \dots t_n}^{-1}(A_1 \times \dots \times A_n)).$$

where $\pi_{t_1 \dots t_n} : X^{<0,1>} \rightarrow X^{\{t_1 \dots t_n\}}$ denotes the canonical projection.

Theorem 1. $\vec{\mu}$ extends uniquely to a vector measure

$$\vec{\mu} : \mathcal{B}(\mathcal{T}) \rightarrow E.$$

Example. It is not true in general, that $\vec{\mu}$ lives on $\mathcal{T}_{\text{meas}}$!
For an arbitrary Baire probability on X , put

$$\mu(\pi_{t_1 \dots t_n}^{-1}(A_1 \times \dots \times A_n)) = \prod_{i=1}^n \nu(A_i) \quad \text{and extend this}$$

according to the Kolmogorov theorem.

The resulting probability on $X^{<0,1>}$ gives rise to a measure $\tilde{\mu}$, which is, as can be easily shown, supported by a single element of $\mathcal{T}$, namely by $\lambda \otimes \nu$, where λ denotes the Lebesgue measure!

Notation. Denote by

$$I_\delta = \{(t, s) \in <0, 1> \mid |t-s| < \delta\}$$

$$\lambda_\delta = \frac{1}{2\delta} \lambda$$

$$I_\delta^n = \{(t_1 \dots t_n) \in <0, 1>^n \mid |t_i - t_j| < \delta\}$$

$$\lambda_\delta^n = \left(\frac{1}{2\delta}\right)^{n-1} \lambda$$

Theorem 2. $\tilde{\mu}$ has the support in $\mathcal{T}_{\text{meas}} \iff$ for each $\varepsilon > 0$ and A open the following holds:

$$\delta \searrow 0 \implies \lambda_\delta \left\{ (t, s), \left\| \tilde{\mu}(\pi_t^{-1}(A) \Delta \pi_s^{-1}(A)) \right\| > \varepsilon \right\} \rightarrow 0.$$

Example. The latter condition holds e.g. in the case, when

$$\left\| \tilde{\mu}(\pi_s^{-1}(A) \Delta \pi_t^{-1}(A)) \right\| \xrightarrow{s \rightarrow t} 0 \quad \text{holds for almost all } t \in <0, 1>.$$

This corresponds to the a.e. stochastic continuity of the process.

Theorem 3. $\tilde{\mu}$ has the support in $\mathcal{T}_{\text{cont}} \iff$ for each $\varepsilon > 0$ and A open the following holds:

$$\delta \searrow 0 \implies \lambda_\delta^n \left\{ (t_1 \dots t_n), \left\| \pi_{t_1}^{-1}(A) \setminus \pi_{t_1 \dots t_n}^{-1}(A \times \dots \times A) \right\| > \varepsilon \right\}$$

holds for some $i \} \rightarrow 0$ uniformly with respect

to $n \in \mathbb{N}$.

Note. For Markov processes, this gives rather weak results (a.e. continuity instead of the nonexistence of the discontinuities of 2. kind).

In order to prove an analogy of Th. 3 for trajectories without

discontinuities of 2. kind, we should use some more subtle Markovian arguments.

Finally, let us show that if $\tilde{\mu}$ has the support in $\mathcal{T}_{\text{meas}}$, then it can be viewed as a measure on $X^{<0,1>}$. Choose a partition of $<0,1>$ consisting of all intervals

$$I_n^i = \left\langle \frac{i}{2^n}, \frac{i+1}{2^n} \right\rangle, \quad i=0,1,\dots,2^n-1.$$

For each $t \in <0,1>$ choose $i(n)$ such that $t \in \left\langle \frac{i}{2^n}, \frac{i+1}{2^n} \right\rangle$.

Call $t \in <0,1>$ a Lebesgue point of α_f if there exists $y \in X$ such that for each A open containing y ,

$$\lim_{n \rightarrow \infty} (2^{nP_{I_{i(n)}}}(\alpha)) = 1.$$

Then it can be shown, that there exists a null set $N \subset <0,1>$ such that the following is true:

For each $t \in <0,1> \setminus N$, t is the Lebesgue point of $\tilde{\mu}$ almost all trajectories and if we fix some $x_0 \in X$ and put

$$\begin{aligned} \mathcal{F}(t) &= y && \text{whenever } t \text{ is a Lebesgue point of } \alpha_f, \\ \mathcal{F}(t) &= x_0 && \text{otherwise} \end{aligned}$$

then the map

$\{\alpha_f \rightsquigarrow \mathcal{F}\} : \mathcal{T}_{\text{meas}} \rightarrow X^{<0,1>}$ is Baire measurable and the image of $\tilde{\mu}$ coincides with μ on $\mathcal{B}(X^{<0,1>} \setminus N)$.

References

- [1] Gihman-Skorochod: The theory of stochastic processes. Part I.