On a Michael's conjecture concerning the Lindelõf property in the Cartesian products

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It is known that if Y is a hereditarily Lindelõf space and X is a separable metric space or a Lindelõf complete in the sense of Čech space or a C - scattered space then $Y \times X^{\aleph_0}$ is Lindelõf. Let us recall that X is a C - scattered space if every closed subset F of X contains compact set with non-empty interior with respect to F. The first result, mentioned above, is due to S. Willard, second one due to Z. Frolík, third one due to K. Alster.

Michael conjectured that if $Y \times X$ is Lindelõf for every hereditarily Lindelõf space Y then $Y \times X^{\aleph_0}$ is Lindelõf for every hereditarily Lindelõf space Y.

The answer to the Michael's conjecture is a negative one provided that the condition (*) holds.

The condition (*) says that

(*) there exists an uncountable coanalytic subset of the Cantor set which does not contain uncountable compact subsets.

Gõdel and P.S. Novikov proved that (*) holds under the Gõdel's axiom constructibility. L. Bukovský, D.A. Martin, R.M. Solovay and P. Vopěnka defined a model of set theory such that $\aleph_1 < 2^{\aleph_0}$ and every subset of the Cantor set of cardinality $\aleph_1$ is coanalytic. R.M. Solovay proved that if a measurable number exists then every coanalytic set contains the Cantor set.

Under the condition (*) I have obtained the following two examples.

Example 1 (*) . There exists X such that for every hereditarily Lindelöf space Y and every natural number n the product $Y \times X^n$ is Lindelöf but $X^{\aleph_0}$ is not.

Example 2 (*) . There exist a separable metric space M and a space Z such that for every Lindelöf space Y and every natural number n the products $Y \times Z^n$ and $Z^{\aleph_0}$ are Lindelöf but $M \times Z^{\aleph_0}$ is not.

Let me finish with the following three problems.

(1) Let $Y \times X$ be a Lindelöf space for every Lindelöf space Y. Is it true that $X^{\aleph_0}$ is Lindelöf.

(2) Let $Y \times X$ be a Lindelöf space for every hereditarily Lindelöf space Y. Is it true that X^2 is Lindelöf.

(3) Is it possible to obtain Example 1 and 2 without set theoretical assumptions.