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Let (X,r) be a metric space, and let $d(A)$ denote the diameter of A . For every positive real number p , a Borel measure m_r^p on X may be defined by the following formula:

$$m_r^p(B) = \lim_{\varepsilon\to 0} \inf \left\{ \sum_{i=1}^{\infty} d^p(A_i) \;\middle|\; \bigcup_{i=1}^{\infty} A_i \supseteq B \;,\; d(A_i)<\varepsilon \right\}$$

m_r^p is called the p-dimensional Hausdorff measure on (X,r). This concept developed by Hausdorff [1] in 1918 has an intuitive geometric meaning. Take $p=1$. $d(A)$ might be called the lenght of A (it is the same for an interval on the real line). The 1-dimensional measure of a Borel set B is approximated in the above definition by the sum of the "lenghts" of small sets which are needed to cover B.
To get Lebesgue measure of a set B in the plane we have to cover B by small squares or circles and to add the area of these sets, the area of A now given by $d^2(A)$ $(p=2)$.

The concept of Hausdorff measure has been neglected in recent time. We emphasize its importance by showing the simple but astonishing fact that every locally finite diffuse (i.e. $m(\{x\})=0$ for all x) measure m on R^n being positive on open sets is an n-dimensional Hausdorff measure with respect to a certain metric compatible with Euclidean topology. Note that every Hausdorff measure is diffuse by definition.

Example Every locally finite diffuse Borel measure m on R is the 1-dimensional Hausdorff measure generated by the pseudometric $r(x,y) = m([x,y[)$. If m is positive on all intervals, r is a metric compatible with Euclidean topology.

Proof: Since $d(A)= m([\inf A,\sup A[) \geq m(A)$, $m_r^1(B) \geq m(B)$ for all B . If B is an interval, there are coverings of B by disjoint intervals of arbitrary small diameter. m and m_r^1 agree on intervals, thus on all Borel sets.

Remark: This shows that Hausdorff measures generated by topologically equivalent metrics need not be absolutely continuous (there are diffuse locally positive finite Borel measures on R singular to Lebesgue measure).
On the other hand, we can show that every Borel measure m on a locally compact metric space (X,r) given by $m(B) = {}_B\int f\, dm_r^p$ where f is a positive continuous function, is the p-dimensional Hausdorff measure with respect to a metric r' topologically equivalent to r . r' is given by the "line integral" of f , that is

$$r'(x,y) = \inf\left\{ \sum_{i=1}^{n-1} \frac{f(c_i) + f(c_{i+1})}{2} \cdot r(c_i, c_{i+1}) \;\middle|\; c_i \in X, c_1 = x, c_n = y \right\}$$

Proposition Let m be a finite Borel measure on Cantor space $D = \{0,1\}^N$ or a locally finite measure on $D - \{o\}$, and let $m(U) > 0$ for every open U . Then for every $p > 0$, m is a p-dimensional Hausdorff measure with respect to a metric on D compatible with the product topology.

Proof: Let r be a metric on D generating the product topology. Let $\underline{P}_n$, n=1,2,... be a sequence of partitions of D into clopen subsets, such that $\underline{P}_{n+1}$ is a refinement of $\underline{P}_n$ for every n and that $d(U_n) < \frac{1}{n}$ for $U_n \in \underline{P}_n$. For two different points x,y of D let n(x,y) be the smallest n for which $\underline{P}_n$ separates x and y and let $r'(x,y) = m(U(x,y))$ where U(x,y) is the member of $\underline{P}_{n(x,y)-1}$ containing x and y . It is easy to see that r' is an ultrametric, that is, it satisfies $r'(x,y) \leq \max(r'(x,z), r'(y,z))$ for all x,y,z . The topology generated by r' is the product topology since it has the open base consisting of all sets of the $\underline{P}_n$. Now $m_{r'}^1 \geq m$ follows from $d'(A) = \min\{m(U) \mid U \supseteq A,\ U \in \bigcup_n \underline{P}_n\} \geq m(A)$. For every compact set B and every $\delta > 0$ there is a neighborhood U of B with $m(U-B) < \delta$. Then $r(B, D-U) = \varepsilon > 0$, and for every n with $\frac{1}{n} < \varepsilon$ the sets of $\underline{P}_n$ intersecting B form a disjoint covering of B with union smaller than U. Thus m and $m_{r'}^1$ agree on compact sets, hence on Borel sets. Since r' is an ultrametric, $(r')^{\frac{1}{p}}$ is a metric for every positive p, and $m = m_{r'}^1$ is the p-dimensional Hausdorff measure with respect to $(r')^{\frac{1}{p}}$.

Proposition Let $\bar{m}$ be a finite or σ-finite non-atomic measure on a separated and separable Borel space $(X, \mathcal{A})$ and $p > 0$. There is a metric r on X with $\bar{m} = m_r^p$.

Proof: All separable separated Borel spaces are isomorphic. There is a one-to-one mapping f from X onto D inducing an isomorphism between $\mathcal{A}$ and the Borel δ-algebra of D. In the δ-finite case $f: X \to D-\{o\} = D \times N$ may be chosen in such a way that $m = f \cdot \bar{m}$ becomes locally finite. Let r' be the metric constructed above from the measure $m = f \cdot \bar{m}$ on D. Then $r(x,y) = (r'(f(x),f(y)))^{1/p}$ yields the desired metric on X.

<u>Proposition</u> Let m be a locally finite diffuse Borel measure on R^n being positive on every open set. Then m is the n-dimensional Hausdorff measure with respect to a certain metric r' compatible with Euclidean topology.

Proof: Oxtoby and Ulam [2] proved that there is a homeomorphism h from R^n onto R^n for infinite m and onto $]0,k[^n$ for finite m, such that $m(B)$ equals the Lebesgue measure $\lambda(h(B))$ for all Borel sets B. Since $\lambda = m_r^n$ where r denotes the max-metric on R^n we have only to put $r'(x,y) = r(h(x),h(y))$.

<u>Remark:</u> Obviously, this statement remains valid for diffuse finite Borel measures on (measurable) subsets of R^n. It may also be generalized to manifolds, and with some modification to arbitrary separable metric spaces.

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<u>References</u>

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