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1. Two Observations

Let K be a compact Hausdorff space and $A \subset K$ a closed subset. By I_A we denote the closed ideal of all $f \in CK$ which vanish on A.
If X is any Banach space and I a closed subspace of X, then for $x \in X$, $P_I(x)$ means the set of best approximation from I to x, i.e. $P_I(x) = \{y \mid y \in I, d(x,I) = \|x-y\|\}$.
We observe that

1. I_A is a complemented ideal iff A is clopen (this is well-known)
2. $P_{I_A}(f)$ is a ball for every $f \in CK$ iff A is clopen (this can easily be proved).

In the sequel we will present some results which can be thought of as a general Banach space formulation of these two observations.

2. M-ideals

Let X be a Banach space and I a closed subspace of X.
We say that

a) I is an M-summand if there is a closed subspace $I^\perp$ of X such that X is the L^∞-sum of I and $I^\perp$.

b) I is an L-summand if there is a closed subspace $I^\perp$ of X such that X is the L^1-sum of I and $I^\perp$.

c) I is an M-ideal if I^π, the annihilator of I in X', is an L-summand.

Examples:

1. The M-ideals in CK are precisely the spaces I_A. I_A is an M-summand iff A is clopen.
2. More generally: in a C^*-algebra the M-ideals are precisely

the closed two-sided ideals .

We note that for M-ideals I the sets $P_I(x)$ are "great" in general: $P_I(x)$ spans I for every $x \notin I$. (For more details we refer the reader to E. Behrends: "M-Structure and the Banach-Stone theorem", Lecture Notes in Mathematics 613, Springer-Verlag).

3. The M-complement of an M-ideal

Let I be an M-ideal of the Banach space X . By $I^{\perp}$ (= the M-complement of I) we mean the collection of all $y \in X$ such that $\| x+y \| = \max\{ \| x \| , \| y \| \}$ for every $x \in I$.

Theorem: For $x \in X$ the following are equivalent

(i) $x \in I^{\perp}$

(ii) $p(x) = o$ for every $p \in (I^{\pi})^{\perp}$

(iii) $p(x) = o$ for every extreme functional which does not vanish on I .

(iv) $P_I(x)$ is the ball with radius $d(x,I)$ and center 0 in I

(v) $P_I(x) = - P_I(x)$

It follows that $I^{\perp}$ is a closed subspace of X which easily implies that I is an M-summand of $I + I^{\perp}$ (this space is in fact the greatest subspace Y of X such that I is an M-summand of Y).
The elements of $I + I^{\perp}$ may be characterized as follows:
$x \in I + I^{\perp}$ iff $P_I(x)$ is a ball iff $P_I(x)$ is symmetric (i.e. there is an $x_o \in P_I(x)$ such that $x_o + y \in P_I(x)$ implies $x_o - y \in P_I(x)$ for $y \in I$) .

Corollary (Evans 1974): Let I be a closed subspace of X . Then I is an M-summand iff the following intersection property holds: $\cap D_i \cap I \neq \emptyset$ for every family (D_i) of closed balls such that $\cap D_i \neq \emptyset$ and $D_i \cap I \neq \emptyset$ for every i .

<u>Proof</u>: It follows from the intersection property that $P_I(x)$ is a closed ball for every x so that $I + I^{\perp} = X$.

<u>Corollary</u>: Let I be an M-ideal in X . Then I is an M-summand iff all $P_I(x)$ are symmetric iff all $P_I(x)$ are closed balls. We note that this corollary contains the two observations of the introduction as a special case.

4. The case of C^*-algebras

It can be shown that, if X is a C^*-algebra, the sets $I^{\perp}$ are also M-ideals. It has been pointed out that $I^{\perp}$ is just the set $\{x \mid xy = yx = o$ for every $y \in I\}$ which is denoted by $\{o\} : I$ in the theory of C^*-algebras. This gives rise to some natural generalizations to arbitrary Banach spaces of the notion of "quotient ideals" which we omit to describe here .