Aleksander Błaszczyk; Andrzej Szymański

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SOME NON-NORMAL SUBSPACES OF THE ČECH-STONE COMPACTIFICATION OF A DISCRETE SPACE

by

A. Błaszczyk and A. Szymański

In this note we shall present some methods for constructions of non-normal subspaces of $\beta\kappa$ - the Čech-Stone compactification of a discrete space of cardinality κ, or of $\omega^* = \beta\omega - \omega$, or of $U(\kappa)$ - the space of all uniform ultrafilters on κ.

Firstly we shall present a method using Hausdorff's gaps.

By a γ-tower on κ we mean an indexed family $\{T_\alpha : \alpha < \gamma\}$ of subsets of κ such that $|T_\alpha - T_\beta| < \kappa$ and $|T_\beta - T_\alpha| = \kappa$ for all $\alpha < \beta < \gamma$.

Lemma 1. If κ is regular and $\{T_\alpha : \alpha < \kappa^+\}$ is a κ^+-tower on κ, then there are κ^+-towers $\{A_\alpha : \alpha < \kappa^+\}$ and $\{B_\alpha : \alpha < \kappa^+\}$ on κ such that:

(1) $A_\alpha \cup B_\alpha = T_\alpha$ and $A_\alpha \cap B_\alpha = \emptyset$ for $\alpha < \kappa^+$.

(2) There is no $C \subset \kappa$ such that $|A_\alpha - C| < \kappa$ and $|C \cap B_\alpha| < \kappa$ for all $\alpha < \kappa^+$

The families $\{A_\alpha : \alpha < \kappa^+\}$ and $\{B_\alpha : \alpha < \kappa^+\}$, as in Lemma 1, form a Hausdorff's gap on κ. With the help of these families we will construct two disjoint closed subsets E,F of the space $T = \bigcup\{cl_{\beta\kappa} T_\alpha : \alpha < \kappa^+\}$ which can not be separated in $\beta\kappa$. Namely, we set $E = \bigcup\{cl_{\beta\kappa} A_\alpha \cap U(\kappa) : \alpha < \kappa^+\}$ and $F = \bigcup\{cl_{\beta\kappa} B_\alpha \cap U(\kappa) : \alpha < \kappa^+\}$. Thus T is an open and non-normal subspace of $\beta\kappa$. It turns out that if I is a P-point ideal on κ (see [T]) and $2^\kappa = \kappa^+$, then there is a κ^+-tower $\{T_\alpha : \alpha < \kappa^+\}$ on κ such that $\bar{I} = \bigcup\{cl_{\beta\kappa} D : D \in I\} = \bigcup\{cl_{\beta\kappa} T_\alpha : \alpha < \kappa^+\}$. Hence

Theorem 1. If κ is regular, $2^\kappa = \kappa^+$ and I is a P-point ideal on κ, then I is not normal.

This theorem improves a similar result by E. van Douwen [vD] stated for normal ideals on regular $\varkappa$.

Let $\{\lambda_\alpha : \alpha < \omega_1\}$ be the order preserving indexing of the limit ordinals in ω_1 and let NL be the set of non-limits ordinals in ω_1.

Lemma 2. There are families $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$ such that:

(o) $|A_\alpha - A_\beta| + |B_\alpha - B_\beta| < \omega$ for $\alpha < \beta < \omega_1$,

(i) $A_\alpha \cup B_\alpha = \lambda_\alpha \cap NL$ and $A_\alpha \cap B_\alpha = \emptyset$ for $\alpha < \omega_1$,

(ii) there is no $C \subset \omega_1$ such that $|A_\alpha - C| < \omega$ and $|B_\alpha \cap C| < \omega$ for all $\alpha < \omega_1$

The families $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$, as in Lemma 2, also form a special kind of Hausdorff's gaps on ω_1. With the help of these families we shall show

Theorem 2. If $\{T_\alpha : \alpha < \omega_1\}$ is an ω_1-tower on ω, then $T^* = \bigcup\{cl_{\beta\omega}T_\alpha \cap \omega^* : \alpha < \omega_1\}$ is a non-normal space.

Proof. The sets $E = \bigcup\{Bd \bigcup\{L_\xi : \xi \in A_\alpha\} : \alpha < \omega_1\}$ and $F = \bigcup\{Bd \bigcup\{L_\xi : \xi \in B_\alpha\} : \alpha < \omega_1\}$, where $L_\xi = cl_{\beta\omega}T_\xi - cl_{\beta\omega}T_{\xi-1}$ for $\xi \in NL$, are disjoint closed subsets of T^* which can not be separated in T^*.

It is easy to see that if the continuum hypothesis, CH, holds and $p \in \omega^*$, then $\omega^* - \{p\}$ contains, as a closed subset, a non-normal space T^* for some ω_1-tower $\{T_\alpha : \alpha < \omega_1\}$ on ω. Hence

Corollary (see [G],[R],[W]). (CH). The space $\omega^* - \{p\}$ is not normal for each $p \in \omega^*$.

The details of the proofs of all above results can be found in our paper [BS]. Besides, we shall present, with details, how the removal of some points p from $U(\varkappa)$ gives non-normality of $U(\varkappa) - \{p\}$.

A set A contained in a space X is called strongly discrete if the points of A can be simultaneously separated by disjoint open subsets of X. Note that countable discrete subsets of regular spaces are strongly discrete.

<u>Theorem 3.</u> If κ is regular and A is a strongly discrete subset of the space $U(\kappa)$ of cardinality $\leq \kappa$, then $U(\kappa)-\{p\}$ is not normal, for each $p \in clA-A$.

Proof. Let $p \in clA-A$. For each $a \in A$ let $D_a \subset \kappa$ be such that $D_a \notin p$, $D_a \in a$ and $|D_a \cap D_b| < \kappa$ whenever $a \neq b$. Such sets D_a exist, since $A \subset U(\kappa)$ is strongly discrete. Let us set $\xi = \{B \subset A\colon p \in clB\}$. Note that ξ is an ultrafilter on A. Now we put $F = \bigcap\{cl\bigcup\{D_a^*\colon a \in B\}\colon B \in \xi\}$, where $D_a^* = clD_a \cap U(\kappa)$. Clearly, F is closed in $U(\kappa)$ and $p \in F$.

<u>Claim.</u> $F \cap clA = \{p\}$. Assume otherwise and let $q \neq p$ be such that $q \in F \cap clA$. There is a $B \in \xi$ such that $q \notin clB$. Since $|A| \leq \kappa$, there is a $C \subset \kappa$ such that $|D_a - C| < \kappa$ for each $a \in B$ and $|D_a \cap C| < \kappa$ for each $a \notin B$ (see [CN]). Hence $F \subset cl_{\beta\kappa}C \cap U(\kappa)$ and $q \notin cl_{\beta\kappa}C \cap U(\kappa)$; a contradiction.

By the Claim, the sets $F-\{p\}$ and $clA-\{p\}$ are disjoint in $U(\kappa)-\{p\}$. Clearly, they are also closed in $U(\kappa)-\{p\}$. It remains to show that $F-\{p$ and $clA-\{p\}$ can not be separated in $U(\kappa)-\{p\}$ by open sets.

Assume otherwise, and let U,V be disjoint open subsets of $U(\kappa)-\{p\}$ containing $F-\{p\}$ and $clA-\{p\}$, respectively. For each $a \in A$ let q_a be in $V \cap D_a-\{a\}$ and let $Q = \{q_a\colon a \in A\}$. Then $clQ \cap clA \ni \emptyset$ and $clQ \cap F \neq \emptyset$. Hence $clQ \cap (F-\{p\}) \neq \emptyset$ and therefore $U \cap Q \neq \emptyset$. But this is impossible since $Q \subset V$ and $U \cap V = \emptyset$.

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