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AUTOMATIC CONTINUITY OF OPERATIONAL CALCULI ON ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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In the present exposition, we are concerned with the automatic continuity problem for algebraic homomorphisms between certain topological algebras. To start with a motivating example, let $L(X)$ denote the Banach algebra of all continuous linear operators on some given complex Banach space X. Then $T \in L(X)$ is called generalized scalar, if there exists a continuous homomorphism $\Phi: C^\infty(\mathbb{C}) \to L(X)$ such that $\Phi(1)=I$ and $\Phi(Z)=T$, where Z denotes the identity function on $\mathbb{C}$. It follows from a construction of Dales that the continuity assumption in this definition is not superfluous. Thus one may look for suitable algebraic conditions on the operator T which force every representing homomorphism $\Phi: C^\infty(\mathbb{C}) \to L(X)$ to be continuous. The following result is typical of several theorems in this direction.

1. Theorem. Let Ω denote an open set in $\mathbb{R}$ or $\mathbb{C}$, $k=o,1,\dots,\infty$ and consider an algebrahomomorphism $\Theta: C^k(\Omega) \to L(X)$ such that $\Theta(1)=I$. Then for $T:=\Theta(Z)$ the following assertions are equivalent:

(a) T is a generalized scalar operator.

(b) $\{o\}$ is the only divisible linear subspace of X with respect to T, i.e. every linear subspace Y of X with $(T-\lambda I)Y=Y$ for all $\lambda \in \mathbb{C}$ has to be $=\{o\}$.

(c) The restriction of Θ to $C^{2k+1}(\Omega)$ is continuous with respect to the $C^{2k+1}(\Omega)$ - topology.

Here, the implication (c)⇒(a) is obvious, while (a)⇒(b) follows from the structure of the spectral maximal spaces of a generalized scalar operator due to Vrbová. The remaining implication (b)⇒(c) requires the general theory to be sketched below. We first indicate some typical applications of theorem 1, where B denotes an arbitrary commutative Banach algebra over $\mathbb{C}$. The author has been informed by Professor Dales that in assertion (ii) the order $2k+1$ is best possible - at least if one assumes the continuum hypothesis.

2. Corollary. (i) If $T\in L(X)$ has no divisible linear subspace different from $\{o\}$, then every spectral distribution for T is automatically continuous.
(ii) If $\Theta: C^k(\Omega) \to B$ is an algebrahomomorphism which is continuous on $C^\infty(\Omega)$, then Θ is even continuous on $C^{2k+1}(\Omega)$.
(iii) If the radical of B is nil, then every algebrahomomorphism $\Theta: C^k(\Omega) \to B$ is continuous on $C^{2k+1}(\Omega)$.

We now turn to an automatic continuity result for generalized local linear operators which applies not only to homomorphisms, but also to classical local linear operators between spaces of functions or distributions and to linear mappings intertwining, for instance, two given generalized scalar operators. Let X and Y be Hausdorff topological vector spaces. Then the separating space of a linear operator $\Theta: X\to Y$ is defined to be

$$\mathfrak{S}(\Theta) := \{y\in Y:\ \exists \text{ a net } (x_\alpha)_\alpha \text{ in } X \text{ with } x_\alpha \to o \text{ and } \Theta x_\alpha \to y\} \ .$$

This space is a useful tool in automatic continuity theory since $\mathfrak{S}(\Theta)=\{o\}$ forces Θ to be continuous as soon as a closed graph theorem is valid for linear mappings from X to Y. Given a regular topological space Ω, a mapping $\mathfrak{C}$ from the family $\mathfrak{F}(\Omega)$ of all closed subsets of Ω into the family $\mathfrak{S}(X)$ of all closed linear subspaces of X is called a precapacity if $\mathfrak{C}$ is monotone with $\mathfrak{C}(\emptyset)=\{o\}$.

3. Theorem. Consider an (F)-space X and a precapacity $\mathfrak{C}_X$ from $\mathfrak{F}(\Omega)$ into $\mathfrak{S}(X)$ such that

$$X = \mathfrak{C}_X(\bar{U}) + \mathfrak{C}_X(\bar{V}) \ \text{ for all open } U,V\subset\Omega \text{ with } \Omega = U\cup V \ .$$

Further, let Y denote a topological vector space with a fundamental sequence of bounded sets and let $\mathfrak{C}_Y$ be a precapacity from $\mathfrak{F}(\Omega)$ into $\mathfrak{S}(Y)$ such that

$$\mathfrak{C}_Y\Big(\bigcap_\alpha F_\alpha\Big) = \bigcap_\alpha \mathfrak{C}_Y(F_\alpha) \quad \text{for all } F_\alpha \in \mathfrak{F}(\Omega) \ .$$

Then for every linear operator $\Theta: X\to Y$ being local in the sense that

$$\Theta\,\mathfrak{C}_X(F) \subset \mathfrak{C}_Y(F) \quad \text{for all } F\in\mathfrak{F}(\Omega) \ ,$$

there exists a finite subset Λ of Ω such that $\mathfrak{S}(\Theta)\subset\mathfrak{C}_Y(\Lambda)$, and Θ is closed on $\mathfrak{C}_X(F)$ for all $F\in\mathfrak{F}(\Omega)$ with $F\cap\Lambda=\emptyset$.

This theorem is proved via an appropriate gliding hump procedure and is closely related to the automatic continuity of all causal and translation-invariant linear operators in system theory. As an easy consequence we obtain the following result due to Peetre concerning the abstract characterization of differential operators. It should be noted that in this example the continuity assertion ceases to be true in general if the range space $C(\Omega)$ is replaced by $\mathfrak{D}'(\Omega)$. Thus the singularity set Λ occurring in the last theorem need not be empty.

4. Corollary. Let Ω denote an open subset of $\mathbb{R}^n$. Then every linear operator $\Theta:\mathfrak{D}(\Omega)\to C(\Omega)$ with $\operatorname{supp}\Theta f\subset\operatorname{supp} f$ for all $f\in\mathfrak{D}(\Omega)$ is necessarily continuous and hence a differential operator with continuous coefficients.

A suitable combination of theorem 3 with some further automatic continuity theory leads to the following result which in turn is relevant to the proof of the implication (b)$\Rightarrow$(c) in theorem 1.

5. Theorem. Let A denote an admissible (F)-algebra of complex functions on some subset Ω of $\mathbb{C}$, for example $A=C^k(\Omega)$ or $A=C^k(\overline{\Omega})$. And consider an algebraic homomorphism $\Theta:A\to L(X)$ with $\Theta(1)=I$ such that $T:=\Theta(Z)$ has no divisible linear subspace different from $\{o\}$. Then $p(T)\Theta:A\to L(X)$ is continuous for some nonzero polynomial p with complex coefficients.

Let us finally consider the automatic continuity problem for homomorphisms on $C^k(\Omega)$ in the case that no assumption is made on $\Theta(Z)$. Even in this situation it turns out that the homomorphism is continuous on a fairly large part of the algebra $C^{2k}(\Omega)$ with respect to its own topology. By contrast, let us mention an example of Badé and Curtis which shows that a homomorphism on $C^1(\Omega)$ may be discontinuous on every dense subalgebra of $C^1(\Omega)$.

6. Theorem. Consider an open subset Ω of $\mathbb{C}^n$, some $k=o,1,\dots,\infty$ and a topological algebra B with a fundamental sequence of bounded sets. Then for every homomorphism $\Theta:C^k(\Omega)\to B$ there exists a finite set Λ in Ω such that Θ is continuous on the ideal

$$\{f\in C^{2k}(\Omega)\ :\ f\equiv o \text{ in some neighbourhood } U(f) \text{ of } \Lambda\}$$

in $C^{2k}(\Omega)$ with respect to the $C^{2k}(\Omega)$-topology.

The last theorem of this note deals with the problem of replacing a discontinuous functional calculus by a suitable continuous one. If an operator $T\in L(X)$ admits a discontinuous operational calculus on some admissible topological algebra A, the problem arises to construct a continuous operational calculus on an appropriate topological algebra which is algebraically as well as topologically sufficiently near to A. For simplicity we restrict ourselves to the case $A=C^k(\Omega)$ here. Then, in the context of several complex variables, the solution of this problem reads as follows.

<u>7. Theorem</u>. Let $\Omega\subset\mathbb{C}^n$ be open and assume that $T_1,\dots,T_n\in L(X)$ admit a functional calculus on $C^k(\Omega)$ for some $k=0,1,\dots,\infty$ in the sense that there exists an algebraic homomorphism $\Theta:C^k(\Omega)\to L(X)$ with $\Theta(1)=I$ and $\Theta(Z_j)=T_j$ for $j=1,\dots,n$, where $Z_1,\dots,Z_n$ denote the coordinate functions on Ω. Then there exists a continuously embedded subalgebra C of $C^k(\Omega)$ such that the following conditions are satisfied:

(a) C is the countable inductive limit of (F)-algebras.

(b) C is admissible.

(c) $T_1,\dots,T_n$ admit a continuous functional calculus on C.

The proof of this theorem is based on some kind of interpolation between the given (possibly discontinuous) non-analytic functional calculus and the ordinary analytic functional calculus in several variables. Another essential ingredient is theorem 3. All results of this exposition have been obtained in joint work with Ernst Albrecht. Further information can be found in the following references.

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