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A note concerning the "Feynman integrability" of sets of trajectories

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For the information on the subject of Feynman integral, see [1] and [2] .
There were, on these Schools, some lectures concerning this theme, too.
The effort in this area is mainly done in the direction of finding some reasonable classes of "Feynman integrable" functions.

Perhaps the most complete treatment of this subject is the Albeverio-Hoegh Krohn theory [2] .

Another approach to the subject is presented in a series of articles of Cameron, Storvick, Johnson and Skoug (see e.g. [3]).

A typical example of a "Feynman integrable" function on trajectories is a multiplicative function of the type

$$(1) \qquad \left\{ x \rightsquigarrow e^{i \int_a^b U(x(t),t)\,dt} \right\} : X^{\langle a,b \rangle} \to C$$

where $X = R^j$, j=1,3, etc. In the following, a=0 and b=1.
The potential U is supposed to be sufficiently "regular".

On the contrary, no examples of "Feynman integrable" sets of trajectories are explicitly shown in these theories.
In fact, the example presented there is the only simple example of "Feynman integrable" set known to the author of this note.

We choose a quite general approach to the Feynman integral:

given any semigroup $T = \{e^{tA}\}$ of operators on some $L^p(X, \mu)$, define, for each cylindrical rectangle of the type

$$\bar{\Omega} = \mathcal{J}^{-1}_{t_o,\dots,t_n}(\Omega_o \times \dots \times \Omega_n) \in \mathcal{B}(X^{\langle 0,1 \rangle})$$

$$(0 = t_o \leq t_1 \leq \dots \leq t_n = 1)$$

the Feynman operator integral by the formula

$$(2) \qquad \vec{\mu}_T(\bar{\Omega}) = \chi_{\Omega_n} \circ e^{(t_n - t_{n-1})A} \circ \chi_{\Omega_{n-1}} \circ \dots \circ e^{t_1 A} \circ \chi_{\Omega_o}$$

where χ_{Ω_i} denotes the operator of multiplication by a characteristic function of Ω_i .

It can be shown, that, roughly speaking, there is no vector measure extending (2) except in the case of stochastic semigroup (see [4]).

But, of course, the case $A = i\triangle$ (where $\triangle$ is Laplacian, or, more generally, some selfadjoint operator) is of the main interest.

This is the main difficulty in the theory of Feynman "integral".

In these note, we will show the "Feynman integrability" of tubes.

We will, for simplicity, consider only flat tubes, i.e. the sets of trajectories of the type

$$(3) \qquad t_\Omega = \{x \in X^{\langle 0,1 \rangle},\ x(t) \in \Omega\ \forall t \in \langle 0,1 \rangle\}\ , \quad \Omega \in \mathcal{B}(X)\ .$$

Consider any partition $\mathcal{D} = \{t_i\ ,\ 0 = t_o \leq \dots \leq t_n = 1\}$ of the interval $\langle 0,1 \rangle$.

Our main question is:

does (2) converge for $\Omega_i = \Omega$ fixed and $\|\mathcal{D}\| \to 0$?

If this limit exists, it is natural to call it the (operator) Feynman integral of the tube t_Ω .

In an analogous situation of functions (1), there ia a theory of Feynman integral, due to Nelson and others, based on the Trotter formula (5). We choose a similar approach, but the perturbations of semigroups studied there will be different from those studied in the Trotter formula:
namely, we study the convergence of $(\chi_\Omega e^{\frac{A}{n}})^n$.
Now, our main result says.

Theorem.

Let A be a selfadjoint positively definite operator on some $L^2(X,\mu)$.

Consider the quadratic form $\langle A(.),(.)\rangle$ and denote by $\langle\!\langle(.),(.)\rangle\!\rangle$ its Fridrichs extension.

Denote by $\langle\!\langle \quad \rangle\!\rangle_\Omega$ the restriction of $\langle\!\langle \quad \rangle\!\rangle$ to $L^2(\Omega,\mu)$ (i.e. to those functions in $L^2(X)$ with support in Ω) and denote by B the Fridrichs representation of $\langle\!\langle \quad \rangle\!\rangle_\Omega$ (thus, formally, $\langle\!\langle(.),(.)\rangle\!\rangle_\Omega = \langle B(.),(.)\rangle$) .

Suppose further, that the following conditions are satisfied:

(1) $\mathcal{D}(A)\cap\mathcal{D}(B)$ is dense in $L^2(\Omega,\mu)$

(2) B has compact resolvent

(3) $\forall\varepsilon\,\exists\,\delta,\ |||\!|\,f\,|\!|||\leq 1,\ \|\,f.\chi_{\Omega^c}\,\|<\delta \Longrightarrow$
$\exists\, g\in\mathcal{D}\langle\!\langle \quad \rangle\!\rangle_\Omega$ with $\|\,g-f\,\|<\varepsilon$, $|||\!|\,g\,|\!|||<1$

(we denote by $|||\!|\,f\,|\!||| = \langle\!\langle f,f\rangle\!\rangle$, $\|\,f\,\| = \langle f,f\rangle$) .

Then $(\chi_\Omega e^{\frac{i\lambda A}{n}})^n \longrightarrow e^{i\lambda B}$ $(\lambda\in R)$
in measure in the strong operator topology.

Some comments to (1), (2), (3):

If Ω is sufficiently smooth and bounded in R^3, $A = \Delta$

then (1) and (2) obviously hold.

(3) can be shown by use of Green formulas.

Proof of the theorem can be obtained by using the analytic continuation method, the Fatou-Privalov theorem and the following result.

Theorem.

Suppose that the assumptions of the previous thm are satisfied. Then

$$(\chi_{\Omega} e^{-\frac{A}{n}})^n \longrightarrow e^{-B}$$

in the strong operator topology.

Note. I don't known a direct proof of the Theorem.

The details of this note will be published elsewhere.

References

[1] Feynman, Hibbs: Quantum mechanics and path integrals. New York 1965

[2] Albeverio, Hoegh Krohn: Mathematical theory of Feynman integral. Lecture Notes in mathematics. Springer 1976

[3] Johnson, Skoug: Journal Func.Anal. 12, No. 2 (1973), 129-152

[4] Zahradnik: Proc. of the 5th Winter School (1977)

[5] Nelson: Feynman integrals and the Schrödinger equation. Journal of Math.Phys.Vol. 5, No. 3 (1964)