Heinrich von Weizsäcker

A simple example concerning the global Markov Property of lattice random fields

In: Zdeněk Frolík (ed.): Abstracta. 8th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1980. pp. 194--198.

Persistent URL: http://dml.cz/dmlcz/701207

EIGHTH WINTER SCHOOL ON ABSTRACT ANALYSIS (1980)

A simple example concerning the global Markov Property of lattice random fields

Heinrich v. Weizsäcker

1. Notation.

Let T be the vertex set of a countable graph (eg. $\mathbb{Z}^d$, $d \geq 1$). For $\Lambda \subset T$ define the boundary $\partial\Lambda = \{l \in T : l \notin \Lambda$ but l is adjacent to some element of $\Lambda\}$. Let S be a Polish state space. On $\Omega = S^T$ consider the σ-algebras $\mathcal{F}_\Lambda = \{\{\omega : \omega|_\Lambda \in B\} : B \in \text{Borel } (S^\Lambda)\}$. A probability measure P on $\mathcal{F}_T$ determines a "lattice random field".

Definition: P has the *local Markov-Property* if the conditional distributions $P(\cdot,\cdot \mid \mathcal{F}_\Lambda)$ with respect to P satisfy

$$P(A,\omega \mid \mathcal{F}_{\partial\Lambda}) = P(A,\omega \mid \mathcal{F}_{T\setminus\Lambda})$$

for all $A \in \mathcal{F}_\Lambda$ and P-almost all $\omega \in \Omega$, whenever Λ is a *finite* subset of T. If this holds for all *infinite* $\Lambda \subset T$ as well, then P is said to have the *global Markov-Property*. (There are obvious more symmetric reformulations of this definition using conditional expectations.)

2. The Problem

We study the question: When does the local Markov property imply the global one?

First let us remark that for $T = \mathbb{Z}$ the global Markov Property of P is equivalent to saying that P is the law of a (not necessarily homogeneous) Markov chain. Suppose that P describes a "random line", i.e. $\omega(k) = a(\omega)k + b(\omega)$ P-a.e. for all $k \in \mathbb{Z}$ and two real random variables a,b. Then it is easy to see that P in general does not describe a Markov chain, i.e. it does not have the global Markov property. But it has the local Markov Property, since every finite subset Λ of $\mathbb{Z}$ has at least two boundary points k_o, k_1 with, say, $k_o < k_1$; so the values $a(\omega)$, $b(\omega)$ are determined by $\omega|_{\partial\Lambda}$.

But in this example an easy explanation consists in the non-trivial tail behaviour: Given $\omega(k_o)$ the additional information contained in $\omega(k_1)$ is still present in the asymptotic behaviour as $k_1 \longrightarrow +\infty$. Considerations like this suggest the

Problem: Let F_∞ be the tail σ-algebra $\bigcap_{\Lambda \text{ finite}} F_{T\setminus\Lambda}$. Does the local Markov-Property imply the global one, if P/F_∞ is trivial?

The answer is positive if $T = \mathbb{Z}$ and S is countable. ([2], p. 447). The global Markov property has been also established in a number of higher dimensional cases, even for continuous parameter set (an appropriate definition. See [1] and the references there). I am inclined to say that in all these cases the main idea is to verify the hypothesis of the following

<u>Proposition:</u> The local Markov property implies the global one, if for each Λ P-almost all conditional probability measures $P(\cdot,\omega \mid \mathcal{F}_{\partial\Lambda})$ are trivial on $\mathcal{F}_\infty \cap \mathcal{F}_\Lambda$.

One way to prove and to use this proposition is to apply the characterization of triviality on $\mathcal{F}_\infty$ by an extreme point property ([3]).

3. The example.

We construct a field with the local but without the global Markov property which is trivial on $\mathcal{F}_\infty$. It can be interpreted both as an example for $T = \mathbb{Z}^2$ and $S = \{0,1\}$ and (considering the column process) for $T = \mathbb{Z}$ and $S = \{0,1\}^{\mathbb{Z}}$.

Let $(\eta_k)_{k \in \mathbb{Z}}$ be a sequence of independent Bernoulli variables. For $n > 0$ define $S_n := \sum_{k=0}^{n} \eta_k$ mod 2 and $S_{-n} := \sum_{k=0}^{n} \eta_{-k}$ mod 2. For $(m,l) \in \mathbb{Z}^2$ define $\xi(m,l)$ by

$$\xi(m,l) = \begin{cases} \eta_l & \text{if } l > m \geq 0 \text{ or } l < m \leq 0 \\ & \text{or } l = 0,\ m \in \{-1,1\} \text{ or } l = m = \pm 1. \\ S_l & \text{if } l > 1 \text{ and } m \in \{-1,l,l+1\} \\ & \text{or } l < 1 \text{ and } m \in \{-l-1,-l,1\} \\ S_1 & \text{if } l = 1,\ m \in \{-1,2\} \text{ or } m = 2,\ l = o \\ S_{-1} & \text{if } l = 1,\ m \in \{-2,1\} \text{ or } m = -2,\ l = 0. \\ \multicolumn{2}{l}{\text{independent of all other variables}} \\ \multicolumn{2}{l}{\text{if } (m,l) \text{ is not of the above form. (eg. if } m = l = 0)} \end{cases}$$

Thus we get the following picture (a × indicating that the corresponding $\xi(m,l)$ is independent of all others)

$$\begin{array}{cccccccccc}
 & & \vdots & \vdots & \vdots & & & & & \\
 & & \times & S_3 & \eta_3 & \eta_3 & \eta_3 & S_3 & S_3 & \times \\
 & & \times & S_2 & \eta_2 & \eta_2 & S_2 & S_2 & \times & \\
 & & \times & S_1 & \eta_1 & \eta_1 & S_1 & \times & & \\
 & \times & S_{-1} & \eta_0 & \times & \eta_0 & S_1 & \times & & \\
 & \times & S_{-1} & \eta_{-1} & \eta_{-1} & S_{-1} & \times & & & \\
\times & S_{-2} & S_{-2} & \eta_{-2} & \eta_{-2} & S_{-2} & \times & & & \\
S_{-3} & S_{-3} & \eta_{-3} & \eta_{-3} & \eta_{-3} & S_{-3} & \times & & & \\
 & & & & \vdots & \vdots & \vdots & & &
\end{array}$$

It is not difficult to verify that the law of this process has all required properties. (I do not claim that this is a very natural example ...).

A c k n o w l e d g e m e n t: I am very grateful for discussions on this subject with H. Föllmer and M. Scheutzow. From the former I learned about this question and the latter

pointed out a mistake in an earlier version of the example.

References:

[1] Albeverio, S. and Høegh-Krohn, R.: The global Markov Property for euklidean and lattice fields. Physics Letters 84B (1979) n°. 1.

[2] Griffeath, D.: Introduction to random fields. (= Chapter 12 in Kemeny, Snell, Knapp: Denumerable Markov Chains, 2nd. ed.; Berlin: Springer 1976).

[3] Preston, C.: Random fields. Lecture Notes in Math. 534, Berlin: Springer 1976.