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1. Notation.

Let T be the vertex set of a countable graph (eg. $\mathbb{Z}^d$, $d \geq 1$). For $\Lambda \subset T$ define the boundary $\partial\Lambda = \{l \in T : l \notin \Lambda$ but l is adjacent to some element of $\Lambda\}$. Let S be a Polish state space. On $\Omega = S^T$ consider the σ-algebras $\mathcal{F}_\Lambda = \{\{\omega : \omega|_\Lambda \in B\} : B \in \text{Borel}(S^\Lambda)\}$. A probability measure P on $\mathcal{F}_T$ determines a "lattice random field".

Definition: P has the *local Markov-Property* if the conditional distributions $P(\cdot,\cdot \mid \mathcal{F}_\Lambda)$ with respect to P satisfy

$$P(A,\omega \mid \mathcal{F}_{\partial\Lambda}) = P(A,\omega \mid \mathcal{F}_{T\setminus\Lambda})$$

for all $A \in \mathcal{F}_\Lambda$ and P-almost all $\omega \in \Omega$, whenever Λ is a *finite* subset of T. If this holds for all *infinite* $\Lambda \subset T$ as well, then P is said to have the *global Markov-Property*. (There are obvious more symmetric reformulations of this definition using conditional expectations.)

2. The Problem

We study the question: When does the local Markov property imply the global one?

First let us remark that for $T = \mathbb{Z}$ the global Markov Property of P is equivalent to saying that P is the law of a (not necessarily homogeneous) Markov chain. Suppose that P describes a "random line", i.e. $\omega(k) = a(\omega)k + b(\omega)$ P-a.e. for all $k \in \mathbb{Z}$ and two real random variables a,b. Then it is easy to see that P in general does not describe a Markov chain, i.e. it does not have the global Markov property. But it has the local Markov Property, since every finite subset Λ of $\mathbb{Z}$ has at least two boundary points k_o, k_1 with, say, $k_o < k_1$; so the values $a(\omega)$, $b(\omega)$ are determined by $\omega|_{\partial\Lambda}$.

But in this example an easy explanation consists in the non-trivial tail behaviour: Given $\omega(k_o)$ the additional information contained in $\omega(k_1)$ is still present in the asymptotic behaviour as $k_1 \longrightarrow +\infty$. Considerations like this suggest the

<u>Problem:</u> Let F_∞ be the tail σ-algebra $\bigcap_{\Lambda \text{ finite}} F_{T\setminus\Lambda}$. Does the local Markov-Property imply the global one, if P/F_∞ is trivial?

The answer is positive if $T = \mathbb{Z}$ and S is countable. ([2], p. 447). The global Markov property has been also established in a number of higher dimensional cases, even for continuous parameter set (an appropriate definition. See [1] and the references there). I am inclined to say that in all these cases the main idea is to verify the hypothesis of the following

Proposition: The local Markov property implies the global one, if for each Λ P-almost all conditional probability measures $P(\cdot,\omega \mid F_{\partial\Lambda})$ are trivial on $F_\infty \cap F_\Lambda$.

One way to prove and to use this proposition is to apply the characterization of triviality on F_∞ by an extreme point property ([3]).

3. The example.

We construct a field with the local but without the global Markov property which is trivial on F_∞. It can be interpreted both as an example for $T = \mathbb{Z}^2$ and $S = \{0,1\}$ and (considering the column process) for $T = \mathbb{Z}$ and $S = \{0,1\}^{\mathbb{Z}}$.

Let $(\eta_k)_{k \in \mathbb{Z}}$ be a sequence of independent Bernoulli variables. For $n > 0$ define $S_n := \sum_{k=0}^{n} \eta_k \mod 2$ and $S_{-n} := \sum_{k=0}^{n} \eta_{-k} \mod 2$. For $(m,l) \in \mathbb{Z}^2$ define $\xi(m,l)$ by

$$\xi(m,l) = \begin{cases} \eta_l & \text{if } l > m \geq 0 \text{ or } l < m \leq 0 \\ & \text{or } l = 0,\ m \in \{-1,1\} \text{ or } l = m = \pm 1. \\ S_l & \text{if } l > 1 \text{ and } m \in \{-1, l, l+1\} \\ & \text{or } l < 1 \text{ and } m \in \{-l-1, -l, 1\} \\ S_1 & \text{if } l = 1,\ m \in \{-1,2\} \text{ or } m = 2,\ l = o \\ S_{-1} & \text{if } l = 1,\ m \in \{-2,1\} \text{ or } m = -2,\ l = 0. \\ \text{independent of all other variables} & \\ \text{if } (m,l) \text{ is not of the above form. (eg. if } m = l = 0) & \end{cases}$$

Thus we get the following picture (a × indicating that the corresponding $\xi(m,l)$ is independent of all others)

		⋮	⋮	⋮					
		×	S_3	η_3	η_3	η_3	S_3	S_3	×
		×	S_2	η_2	η_2	S_2	S_2	×	
		×	S_1	η_1	η_1	S_1	×		
	×	S_{-1}	η_0	×	η_0	S_1	×		
	×	S_{-1}	η_{-1}	η_{-1}	S_{-1}	×			
×	S_{-2}	S_{-2}	η_{-2}	η_{-2}	S_{-2}	×			
S_{-3}	S_{-3}	η_{-3}	η_{-3}	η_{-3}	S_{-3}	×			
				⋮	⋮	⋮			

It is not difficult to verify that the law of this process has all required properties. (I do not claim that this is a very natural example ...).

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pointed out a mistake in an earlier version of the example.

References:

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