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On metric projections and distance functions in Banach spaces


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We shall consider a real Banach space $X$ and a nonempty closed subset $F \subseteq X$. For $x \in X$ denote by $d_F(x)$ the distance from the point $x$ to the set $F$. The metric projection $P_F(x)$ on the set $F$ is defined as the (possibly) multivalued operator

$$P_F(x) = \{ y \in F; \|x - y\| = d_F(x) \}.$$ 

The set of all $x$ for which $P_F(x)$ contains at least two points will be denoted by $A_F$. The function is termed $\delta$-convex if it is the difference of two convex functions. The hypersurface in $X$ is termed Lipschitz (resp. $\delta$-convex) if it is described by a Lipschitz (resp. Lipschitz $\delta$-convex) function (see [7] or [5] and [6]). The sets $A_F$ was studied e.g. in [2],[4],[3],[5]. If $X$ is a separable strictly convex Banach space then $A_F$ can be covered by countably many Lipschitz hypersurfaces [5]. If $X$ is a separable Hilbert space then there exists [1] a convex $f_F$ (namely $f_F(x) = 1/2 (\|x\|^2 - d_F^2(x))$) such that $P_F(x) \subseteq \partial f_F(x)$. Using a result on the differentiation of convex functions from [6] we immediately obtain the following

**Theorem 1.** Let $X$ be a separable Hilbert space. Then $A_F$ can be covered by countably many $\delta$-convex hypersurfaces.

**Question 1.** Let $A$ be a $\delta$-convex hypersurface in $\mathbb{R}^n$. Does there exist $F$ such that $A \subseteq A_F$?

Note that it is not difficult to prove that a boundary of a convex body in $\mathbb{R}^n$ is a subset of an $A_F$.

Slightly modifying the Asplund's observation concerning the function $f_F$, we can obtain the following theorem.

**Theorem 2.** Let $X$ be a Hilbert space or a finite dimensio-
nal Banach space such that \( \| x \| \in C^2(X - \{0\}) \). Then \( d_F(x) \) is a locally \( \delta' \)-convex function in \( X-F \).

This theorem has the following consequences.

**Theorem 3.** Let \( X \) be finite dimensional and \( \| x \| \in C^2(X - \{0\}) \). Then \( d_F(x) \) is twice differentiable a.e. in \( X-F \).

**Theorem 4.** Let \( X \) be finite dimensional and \( \| x \| \in C^2(X - \{0\}) \). Then \( A_F \) can be covered by countably many of \( \delta' \)-convex hypersurfaces.

**Question 2.** For which \( X \) each \( A_F \) can be covered by countably many of \( \delta' \)-convex hypersurfaces?

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**REFERENCES**


