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THE CAUCHY-RIEMANN EQUATIONS IN ANTICOMMUTING VARIABLES

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Starting from the deep relation between the system of numbers / complex,quaternions,octonions / and extended supersymmetries N = 2,4,8 we obtain the constraints for superfields.The supersymmetric complex and quaternionic Cauchy-Riemann equations are explicitly given.

I. Introduction

It is well known that for the ordinary commuting numbers the extension of the real number is the complex number, the extension of the complex number is the quaternion and the last extension gives the octonion, because the Hurwitz theorem is valid [1].

We can use such extension on the anticommuting numbers to study the connection between this system of anticommuting numbers and extended SUSY +).

Recently geometric KP(n) models become more popular among physicist and play crucial role in gauge theories [2]. These models are known σ models, concretely the CP(n) model and HP(n) model. The CP(n) model is connected with complex fields, which are elements of the coset space $SU(n+1)/SU(n)\times U(1)$, and the HP(n) model is connected with quaternionic fields, which are elements of the coset space $Sp(n+1)/Sp(n)\times Sp(1)$.

It is well known that supersymmetric extension of the CP(n) model is connected with the complex SUSY [3]. Also supersymmetric extension of the quaternionic models was assumed via supercoset approach [4].

Here we shal show the connection between super HP(n) model and quaternionic SUSY or the extended SUSY N = 4. The complex and quaternionic superanalycity will play the role of constraints.

The last step which is not full clear yet is the connection between octonionic super CaP(2) model and N = 8 extended SUSY. This case is of course most interesting because in the N = 8 extended supergravity models one has unified theories of fields incorporating all spins. So as in the extension of numbers also in the N = 8 extended supergravity models it is the last step of the extension, if we want to have spin 2 for graviton [4].

+) SUSY = supersymmetry

2. The complex supersymmetric Cauchy-Riemann equations.

We shall start with the relation between N = 2 extended SUSY and complex anticommuting numbers. We repeat that on the Bose level the coset space $SU(n+1)/SU(n)\times U(1)$ for the CP(n) model can be identified with the complex projective space involves n complex fields $\varphi_i(x) \in SU(n+1)/SU(n)$, i = 1,2,...,n.

The fields $\varphi_i(x)$ are satisfying a constraint

$$\bar{\varphi}_i \varphi^i = 1 \tag{2.1}$$

and two fields related by the U(1) gauge transformation

$$\varphi_i'(x) = e^{i\Lambda(x)} \varphi_i(x). \tag{2.2}$$

It can be also interpreted so that the automorphism group U(1) which preserves the norm of complex numbers is the gauge group.

The U(1) local gauge invariant action of the CP(n) model has the form:

$$S = \frac{1}{2}\int d^2x\, (\overline{D_\mu \varphi_i})(D_\mu \varphi^i) \quad , \tag{2.3}$$

where $D_\mu = \partial_\mu + iA_\mu$ and the Abelian gauge field A_μ has the form

$$A_\mu = \frac{i}{2}\bar{\varphi}_i \overleftrightarrow{\partial}_\mu \varphi^i \tag{2.4}$$

and transforms under (2.2) like $A_\mu' = A_\mu - \partial_\mu \Lambda$.

The super CP(n)model can be obtained via direct supersymmetrization $\varphi_i(x) \longrightarrow \phi_i(x,\theta)$, $D_\mu \rightarrow \nabla_\alpha = D_\alpha - A_\alpha$ in the action (2.3) and constraint (2.2) ,where

$$\phi_i(x,\theta) = \varphi_i(x) + \theta^\alpha \psi_\alpha(x) + \frac{i}{2}\theta^\beta\theta_\beta F(x) \,,$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \not{\partial}\theta_\alpha \,, \quad \not{\partial} = \gamma_\mu \partial^\mu ,$$

$$A_\alpha = \bar{\phi}_i(x,\theta)\,.\, D_\alpha \phi^i(x,\theta) \,, \quad x \equiv (x_0, x_1) \,, \quad \theta \equiv (\theta_1, \theta_2)$$

are the scalar superfield, supercovarint derivative and spinor gauge superfield respectively.

We shall construct the super CP(n) model directly in U(1)- ga uge invariant way using the connection with complex numbers and functions. The gauge group U(1), which preserves the norm of complex numbers, will give the complex SUSY that is equivalent to O(2) real extended SUSY.

We shall combine two real anticommuting variables in the complex one:

$$\theta_\alpha = \theta_\alpha^1 + i\theta_\alpha^2 \quad , \quad \bar{\theta}_\alpha = \theta_\alpha^1 - i\theta_\alpha^2 \tag{2.5}$$

in the full analogy between real and complex numbers.

By analogy with the complex function we shall write a complex superfield $C(x,\theta,\bar\theta) \equiv C(x,\theta^1+i\theta^2,\theta^1-i\theta^2)$.

The SUSY transformation on the complex superspace $(x,\theta,\bar\theta)$ was first defined in two dimensions by M.Ademollo et al.[6]

$$\delta x_\mu = -\frac{i}{2}[\varepsilon\gamma_\mu\bar\theta + \bar\varepsilon\gamma_\mu\theta] \quad , \quad \delta\theta = \varepsilon \quad , \quad \delta\bar\theta = \bar\varepsilon \tag{2.6}$$

and on the superfields $C(x,\theta,\bar\theta)$ acts as follows

$$\delta C = [\varepsilon Q + \bar\varepsilon \bar Q]C,$$

where $Q = \frac{\partial}{\partial\theta} - \frac{i}{2}\not{\partial}\bar\theta \quad , \quad \bar Q = \frac{\partial}{\partial\bar\theta} - \frac{i}{2}\not{\partial}\theta .$

These supercharges anticommute with the covariant derivatives

$$D = \frac{\partial}{\partial\theta} + \frac{i}{2}\not{\partial}\bar\theta = \frac{1}{2}(D^1 - iD^2) \quad , \quad \bar D = \frac{\partial}{\partial\bar\theta} + \frac{i}{2}\not{\partial}\theta = \frac{1}{2}(D^1 + iD^2).$$

We decompose the complex superfield into the real and imaginary part $C(x,\theta,\bar\theta) = A(x,\theta^1\theta^2) + iB(x,\theta^1\theta^2)$.

By analogy with complex functions supersymmetric complex Cauchy-Riemann equations are

$$D^1A = D^2B \tag{2.7a}$$

$$D^1B = -D^2A \tag{2.7b}$$

It means the complex superfield will be an analytic superfield [7], when (2.7a,b) are valid. But it means that the chirality condition [6]

$$\bar D C = 0 \tag{2.8}$$

is the analyticity condition. There this restriction was obtained using the new shifted variable $x' = x - \frac{i}{2}\bar\theta\theta$ (a complex Bose variable) .

The superanalyticity condition actually plays a role of the invariant constraint: $C(x,\theta,\bar{\theta}) = C(x - \frac{i}{2}\bar{\theta}\gamma\theta, \theta)$, what means that the graded Lie algebra in complex SUSY can be realized in a smaller parametric superspace with the complex Bose variable, but independent of the spinor $\bar{\theta}$.

By the same way we can obtain the complex superantianalyticity

$$D\bar{C} = 0.$$

In analogy with the Laplace equation for real and imaginary part of the complex function the superfield equations of motion follows

$$\bar{D}DC = 0 \qquad (2.9)$$

The corresponding action has the form

$$S = \frac{1}{8}\int d^2x\, d^2\theta\, d^2\bar{\theta}\; \bar{C}C \qquad (2.10)$$

If we want to have this action also local U(1)-gauge-invariant we have to use the receipt given in [8]. We have to introduce the vector superfield $V(x,\theta,\bar{\theta})$, which transforms as

$$V \rightarrow V + i(\bar{\Lambda} - \Lambda), \qquad (2.11)$$

under the local U(1) gauge transformation:

$$C \rightarrow e^{i\Lambda}C \quad , \quad \bar{C} \rightarrow e^{-i\Lambda}\bar{C}, \qquad (2.12)$$

where Λ is also an analytical superfield $(\bar{D}\Lambda = 0)$.

It can be shown after [3] that the action

$$S = \frac{1}{8}\int d^2x\, d^2\theta\, d^2\bar{\theta}\,[V - \bar{C}C e^{V}] \qquad (2.13)$$

is supersymmetric, U(1)-gauge-invariant and equivalent the super CP(n) action and super constraint, which is obtained via direct supersymmetrization.

The constraints for ordinary fields in superfield expansion will result from the equation of motion for the vector superfield.

$$\bar{C}C = e^{-V}. \qquad (2.14)$$

In this way we have connected all components in right and left hand side of eq.(2.14) and so vector superfield V acts as a confining force between the scalar superfields.

Complex supersymmetric Cauchy-Riemann equations plays also the role of constraints. Actually for the supersymmetric constraint from [9], which there are assumed "ad hoc", follows from (2.7a,b) and from the anticommutativity of the supercovariant derivatives

$$(D^1D^1 - D^2D^2 + 2iD^1D^2)A = 0 . \tag{2.15}$$

3. The quaternionic supersymmetric Cauchy-Riemann equations

We shall start with the relation between N = 4 extended SUSY in eight dimensional superspace and quaternionic anticommuting numbers. At first we repeat that on the Bose level the space HP(n) can be viewed as the symmetric coset space $Sp(n+1)/Sp(n) \times Sp(1)$

The Sp(1)-local-gauge-invariant action of the HP(n) model has the form

$$S = \frac{1}{2}\int d^4x\,(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})\, tr\,(P\partial_\mu P\partial_\nu P\partial_\alpha P\partial_\beta P) \tag{3.1}$$

where $\mu,\nu,\alpha,\beta = 1,\dots,4$; $P_\mu = P_\mu^+ = P_\mu^2 = \frac{1}{1+u^+u}\begin{pmatrix}1 & u^+\\ u & uu^+\end{pmatrix}$, $u = \begin{pmatrix}u_1\\ \vdots \\ u_n\end{pmatrix}$ and $u_1,\dots,u_n$ are quaternionic coordinates.

The super HP(n) action, which is obtained from (3.1) via direct supersymmetrization $x_\mu \to (x_\mu, \theta_\alpha)$, $\partial_\mu \to D_\alpha$ had to be equivalent to the following SU(2) gauge invariant superaction:

$$S = \frac{1}{32}\int d^4x\, d^4\hat\theta\, d^4\hat{\bar\theta}\, \left(V^1(x,\hat\theta,\hat{\bar\theta}) - \bar{\hat\phi}(x,\hat\theta,\hat{\bar\theta})\hat\phi(x,\hat\theta,\hat{\bar\theta})\, e^{V^1(x,\hat\theta,\hat{\bar\theta})}\right), \tag{3.2}$$

where

$$\hat\phi = \phi_0 + l_k\phi^k, \quad \hat\phi \equiv \hat\phi(x,\hat\theta,\hat{\bar\theta}),$$
$$\hat\theta_\alpha = \theta_\alpha^0 + l_k\theta_\alpha^k, \tag{3.3}$$

V^1 is here the SU(2) gauge vector superfield and for quaternionic units l_k follows:

$$\{l_k, l_m\} = -2\delta_{km},\ [l_k, l_m] = 2\varepsilon_{kmn}l_n,\quad k,m,n = 1, \ ,3 .$$

As in the complex case here can be shown the connection between components fields in the expansion of the quaternionic superfield. For the supercovariant derivatives and supercharges in the quaternionic case is valid:

$$\hat{D}_\alpha = D^0_\alpha + \ell_k D^k_\alpha$$
$$\hat{Q}_\alpha = Q^0_\alpha + \ell_k Q^k_\alpha \qquad (3.4)$$

The supersymmetric quaternionic Cauchy-Riemann equations are given by a concept of Fueter holomorphy in anticommuting variables:

$$\bar{\hat{D}}\hat{\phi} = 0, \qquad (3.5)$$

what is

$$D_0\phi_0 + D_k\phi_k = 0,$$
$$D_0\phi_m - D_m\phi_0 - \varepsilon_{mkn} D_k\phi_n = 0.$$

The condition (3.5), what is the Fueter grassmanian analyticity, which plays a role of constraint:

$$\hat{\phi}(x_\mu, \hat{\theta}, \hat{\bar{\theta}}) = \phi(x_\mu + i\bar{\theta}\gamma_\mu\theta^k\ell_k - \tfrac{i}{2}\bar{\theta}^k\gamma_\mu\theta^l\varepsilon_{klm}\ell_m, \hat{\theta}). \qquad (3.6)$$

It means that quaternionic SUSY can be realized in a smaller superspace, with hypercomplex Bose variable, but independent on $\hat{\bar{\theta}}$.

Similarly we get the Fueter grassmanian antianalyticity condition:

$$\hat{\phi}\overleftarrow{\bar{\hat{D}}} = 0. \qquad (3.7)$$

We remark that also ordinary fields have quaternionic structure. As instructive example is the case when the quaternion is composed from the two complex numbers c_1, c_2 : $\hat{q} = c_1 + \ell_3 c_2$. This case corresponds the physical example of the SUSY dual SU(2) string model [10].

So as in the complex case in [6] the superanalyticity condition gives

$$V_\mu = \overline{DC}\gamma_\mu DC = 0, \qquad (3.8)$$

what is equivalent to the vanishing of the energy-momentum tensor. For the SU(2) string the supercurrent is a vector isotriplet (real) superfield given by

$$V^i_\mu(x, \theta^a_\alpha, \bar{\theta}^a_\alpha) = i\,\overline{D}^{a\alpha}\overline{C}(\sigma^i)_{ab}(\gamma_\mu)_{\alpha\beta}D^{b\beta}C; \quad a, b = 1,2; \; i = 1,2,3,$$

so that the supersymmetric SU(2) invariant Cauchy-Riemann equations give at the classical level $V^i_\mu(x, \theta^a_\alpha, \bar{\theta}^a_\alpha) = 0$.

In such a way quaternionic supersymmetric Cauchy-Riemann equations again play a role of the non-linear constraint which is assumed in ref.[10] "ad hoc".

4. Comments

The open problem in this program of the connection between super sigma models and extended supersymmetries is the possibility of the super CaP(2) model and the connection with N = 8 extended SUSY.

We hope that a SUSY analytical constraints give all constraints in N = 8 supergravity models on 2^{32} ordinary fields.

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