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ON NORM ATTAINING OPERATORS ACTING ON L^1-SPACES

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Given two Banach spaces X and Y we call a linear bounded operator norm attaining if $\|T\| = \|Tx\|$ for some $\|x\| = 1$. In [7] Lindenstrauss raised the problem of deciding when the norm attaining operators are norm dense in the Banach space $B(X, Y)$ of all continuous operators. A review of more recent results in the subject and some open questions can be found in Johnson and Wolfe [6].

The question whether the norm attaining operators are dense in $B(L^1, L^1)$ has been asked by Uhl [9] and answered affirmatively in [4]. Here we comment on and outline the proof of this result.

The proof consists of two parts. Firstly, the operators in $B(L^1, L^1)$ are represented by certain measures on the product space. More specifically, we have the following representation theorem (see [5] ,Prop. 1 and the Theorem):

(1) Let (S_i, Σ_i, m_i) , $i = 1,2$, be finite measure spaces and assume that at least one of the measures is perfect. Then th formula

$$(T_\mu f,h) = \int f(x)h(y)d\mu(x,y) \qquad f\in L^1(m_1),\ h\in L^\infty(m_2)$$

establishes a lattice isomorphism between $B(L^1(m_1), L^1(m_2))$

and the lattice of all σ-additive measures μ on $\Sigma_1 \otimes \Sigma_2$ satisfying $d|\mu|^1/dm_1 \in L^\infty(m_1)$ and $|\mu|^2 \ll m_2$ (here μ^i denote the marginal distributions). Moreover, $\| T_\mu \| = \| d|\mu|^1/dm_1 \|_\infty$.

Note that (1), unlike other known theorems of that kind, is free of any explicit topological assumptions (see [3] and [1]; cf. also [2] where the first part of the proof of Thm.1 is incorrect, no assumptions on the measure spaces being used for the representation by a σ-additive measure on the product space). On the other hand (1) would be false without any measure theoretic assumptions (such as perfectness), although a similar representation by more general set functions is still possible (see [5]). An analogous representation theorem for regular operators on L^p-spaces $(1 < p < \infty)$ has recently been proved by Rębowski [8].

Since every Borel measure on the unit interval is perfect, (1) holds for all such measures. These measures, as is well known, represent (up to Boolean isomorphism) all separable finite measure spaces.

For the second part of the proof we let X, Y be two L^1-spaces. The norm of any T in B(X, Y) is the sup of a sequence Tx_n . The sequences x_n and Tx_n are contained in separable L^1-subspaces and so by general arguments we can restrict ourselves to the separable case: $X = L^1(m_1)$, $Y = L^1(m_2)$, where the m_i's are probability measures on the unit interval. Now it suffices to prove the following measure- theoretic result:

(2) Let μ be a finite (signed) Borel measure on the unit square with $\| d|\mu|^1/dm_1\|_\infty = 1$ and $|\mu|^2 \ll m_2$. For every $\varepsilon > 0$ there exists a measure ν such that $d|\nu-\mu|^1/dm_1 < \varepsilon$, $|\nu|^2 \ll m_2$, $d|\nu|^1/dm_1 \leq 1$, and $(g(y)d\nu(x,y))^1/dm_1 = 1$ on a set B of positive m_1 measure for some function $g \in L^\infty(m_2)$ satisfying $|g| = 1$.

Indeed, if (2) holds than the operator T satisfies $\| T_\nu - T_\mu\| < \varepsilon$ and is norm attaining as $\|T_\nu\| = \langle T_\nu(\chi_B/m_1(B)), g$

For the proof of (2) we choose a measure $\tilde{\mu}$ whose Hahn decomposition $\tilde{F}$, $\tilde{F}^c$ consists of unions of rectangles and such that $|\tilde{\mu}| \leq |\mu|$ and $d|\mu - \tilde{\mu}|^1/dm_1$ is small on a subset C of the set D on which $d|\mu|^1/dm_1$ is close to its sup. Next we pick a subset B of C small enough to be contained in one cell of the partition generated by the projections (into the first axis) of the rectangles constituting the Hahn decomposition of $\tilde{\mu}$. We define $g(y)$ by putting 1 if y is in the projection (into the second axis) of $B\tilde{F}$ and -1 otherwise. It is now not hard to see that the measure

$$d\nu(x,y) = \chi_B(x)(d|\tilde{\mu}|^1/dm_1)^{-1}(x)d\tilde{\mu}(x,y) + \chi_{B^c}(x)d\mu(x,y)$$

satisfies (2). For more details we refer to [4], Thm. 2 (there are minor misprints in [4] on pp. 384 and 385).

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