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Algebras of unbounded operators appear as models in the quantum field mechanics. One of the questions of this theory is the description of the normal states (positive linear functionals) of these algebras. To answer this problem W. TIMMERMANN /5/ has introduced a class of interesting ideals. The operators of these ideals are needed to represent the normal states as trace functionals.

Using some latest results of the theory of locally convex spaces and operator ideals in the main part this lecture we will give a complete description of the geometrical properties of these ideals. In our final remarks we will return to the problem of the representation of linear functionals by the trace. Let us start with some basic notations.

1. The maximal Op^* - algebra $L^+(D)$

To exclude the very pathological cases of operator algebras the considered operators should not be to much unbounded. In /3/ LASSNER has introduced the notion of the maximal Op^*-algebra on a given domain D, which is very convinient from several reasons and which is a straight-forward generalization of the C^*-algebra L(H) of the bounded operators on a Hilbert space H. Let us recall this definition. Let D be a linear dense submanifold in a Hilbert space H. The maximal Op^*-algebra $L^+(D)$ on the domain D is the set of all linear operators X (bounded or not) acting in H and having an adjoint X^* such that $X(D) \subseteq D$ and $X^*(D) \subseteq D$. Of course, $L^+(D)$ is a $*$ - algebra and all operators X $L^+(D)$ are closable and w-continuous in H. Important examples of domains are obtained in the following way. Let X be a positive operator defined in H. Then the set

$$D = \bigcap_{n=0}^{\infty} D(X^n)$$

is the natural domain of the algebra generated by X. The most famous example of this type is given by the operator $X = x^2 - \frac{d^2}{dx^2}$, which is generated by the position and the momentum operators. It is well known and easy to see, that in this case the domain D coincide with the nuclear space $\mathcal{S}(\mathbb{R})$ of the strongly decreasing functions on the real line. Let us return to the general case. The algebra $L^+(D)$ defines in a natural way a <u>topology</u> t <u>on</u> D, which is given by the set of all seminorms

$$p_X(d) := \|Xd\| \quad , \quad d \in D, \quad X \in L^+(D).$$

The domain D is called <u>selfadjoint</u>, if

$$D = \bigcap \{D(X^*) \; : \; X \in L^+(D)\} \; .$$

Under this assumption (D,t) is a complete semireflexive space and its projective spectrum can be identified with the energetic spaces H_X of the X $L^+(D)$, where the norm in H_X is given by $\|d\|_X^2 = \|d\|^2 + \|Xd\|^2$. The imbedding $D \longrightarrow H$ is continuous. Therefore, we have a <u>rigged Hilbert space</u>

$$D \longrightarrow H \longrightarrow D_b' \; .$$

If (D,t) is metrizable then it is a reflexive (F)-space. This is the case in the mentioned above examples. A countable system of seminorms generating the topology t on D is given in this case by $p_n(d) = \|X^n d\|$ for all $n \in \mathbb{N}$.

2. <u>The ideals $\mathcal{A}^{\mathcal{L}}$</u>

We assume in all the following, that D is selfadjoint and metrizable.

A question of some importance in the theory of quantum field statistics is the representation of states by the trace. To solve this problem, TIMMERMANN /5/ has introduced the following ideal in $L^+(D)$:

$\mathcal{N}^{\mathcal{L}} = \{T \in L^+(D) \; : \; \overline{X^* TY}$ is nuclear on H for all $X, Y \in L^+(D)\}$.

For each $T \in \mathcal{N}^{\mathcal{L}}$ the following definition makes sense:

$$w(X) = \text{trace } \overline{XT} = \text{trace } \overline{TX} \; , \quad X \in L^+(D) \; .$$

The linear functional w is called a trace functional. It arises the question, what functionals on $L^+(D)$ have such a trace representation. Before discussing this question in our final remarks, in the main part of this section we will investigate the structure of $\mathcal{N}^{\mathcal{L}}$ or, more generally, of the following ideals:

<u>Definition</u>. Let $\mathcal{A}$ be an operator ideal in L(H). Then

$$\mathcal{A}^{\mathcal{L}} := \{T \in L^+(D) \; : \; \overline{X^* TY} \in \mathcal{A} \text{ for all } X, Y \in L^+(D)\} \; .$$

Proposition 1. $\mathcal{A}^{\mathcal{L}}$ is an $*$ - ideal in $L^+(D)$.

It is easy to see, that the operators $T \in \mathcal{A}^{\mathcal{L}}$ are bounded in fact. Moreover, we have the following characterization.

Theorem 1. An operator $T \in L^+(D)$ belongs to $\mathcal{A}^{\mathcal{L}}$ if and only if T is the restriction of a continuous (antilinear) operator $T_0 : D' \longrightarrow D$ such that all products

$$H \xrightarrow{R} D' \xrightarrow{T_0} D \xrightarrow{S} H$$

belong to $\mathcal{A}$ for all continuous linear operators R, S.

Proof. We restrict us to show, how to extend the operator T. For $h \in H$, $d \in D$ and $X \in L^+(D)$ it holds $|(Xd,Th)| = |(T^*Xd,h)| \leq \leq \|T^*X\|\|d\|\|h\|$. This shows $Th \in D(X^*)$. Because D is assumed to be selfadjoint, it follows $T(H) \subseteq D$. Moreover, T is continuous as a mapping from H in to D. This follows from $p_Y(Th) = = \|YTh\| \leq \|YT\|\|h\|$ for all $Y \in L^+(D)$ and all $h \in H$. The same is true for $T^* : H \longrightarrow D$. The dual operator $T^{*'}$ maps D'_b into $H' \cong H$. We show $T^{*'}(D') \subseteq D$. Let $d \in D$, $d' \in D'$ and $X \in L^+(D)$. It follows $|(Xd,T^{*'}d')| = |\langle T^*Xd,d'\rangle| \leq c \cdot p_Y(T^*Xd) = c\ YT^*Xd \leq c\|YT^*X\|\|d\|$ for some $Y \in L^+(D)$. This shows $T^{*'}d' \in D(X^*)$. But D is selfadjoint. Therefore, we have $T^{*'}(D') \subseteq D$. The operator $T_0 = T^{*'}$ is the wanted extension of T because its restriction to D is T.

This theorem shows, that the operators $T \in \mathcal{A}^{\mathcal{L}}$ are bounded in a very strong sense, especially, they press H into D. It seems to be very difficult to answer the question, wether there are nontrivial (= non finite dimensional) operators in $\mathcal{A}^{\mathcal{L}}$ and what is the structure of the operators $T \in \mathcal{A}^{\mathcal{L}}$. To answer this question let us start with an example of such an operator. Let

$$\mathcal{B}_1 = \{A \in L(H) : A(H) \subseteq D\}, \quad \mathcal{B}_1^* = \{A^* : A \in \mathcal{B}_1\}.$$

If $A \in \mathcal{B}_1$ and $X \in L^+(D)$, then the product XA is closed and defined on H, by the closed graph theorem it must be bounded. Now let $T_1 \in \mathcal{A}(H)$, $A,B \in \mathcal{B}_1$. Then the product $T = AT_1B^*$ belongs to $\mathcal{A}^{\mathcal{L}}$ because of $YTX = YA \cdot T_1 \cdot (X^*B)^* \in \mathcal{A}(H)$ for all $X,Y \in L^+(D)$. It is very surprising that this example covers the general case:

Theorem 2. If $(\mathcal{A}, \alpha)$ is a normed, complete ideal in $L(H)$, then

$$\mathcal{A}^{\mathcal{L}} = \mathcal{B}_1 \cdot \mathcal{A} \cdot \mathcal{B}_1^* .$$

The proof of this theorem is based on a deep result of the theory of locally convex spaces. First of all we need the following proposition.

Proposition 2. Let F be a metrizable locally convex space having a neighbourhood base $\{U_n\}$ such that all spaces F_{U_n} are Hilbert spaces of dimension $\leq \aleph$. Then there is a fundamental system of absolutely convex bounded subsets $\{M_\alpha\}$ such that the spaces F_{M_α} are Hilbert spaces of dimension $\leq \aleph$.

Proof. Let K be any bounded subset of F. Let p_{U_n} be the gauge functional of U_n. We put $c_n = \sup\{p_{U_n}(x) : x \in K\}$ and define

$$M = \{ x \in F : p_M(x)^2 = \sum_{n=1}^{\infty} 2^{-n} c_n^{-1} p_{U_n}(x)^2 \leq 1 \} .$$

This set is bounded and contains K. But p_M satisfies the parallelogramm equation, therefore, F_M must be a Hilbert space of dimension $\leq \aleph_0 \cdot \aleph = \aleph$.

Concerning some notions in the following deep theorem we refer to /4/. In our application to Hilbert spaces the assumed injectivity and surjectivity is not a restricting condition.

Theorem 3. (/1, thm. 7.1.8/). Let $(\mathcal{A}, \alpha)$ be a normed complete operator ideal in the class of Banach spaces which is surjective and injective. Let DF be a barrelled (complete) (DF)-space and F be an (F)-space. If T: DF $\longrightarrow$ F is a linear continuous operator such that the products

$$B_0 \xrightarrow{R} DF \xrightarrow{T} F \xrightarrow{S} B_1$$

belong to $\mathcal{A}$ for all Banach spaces B_0, B_1 and all linear continuous operators R and S, then T admits a linear continuous factorization

$$\begin{array}{ccc} DF & \xrightarrow{\quad T \quad} & F \\ \downarrow & & \uparrow \\ B_2 & \xrightarrow{\quad T_1 \quad} & B_3 \end{array}$$

through Banach spaces B_2, B_3 and an operator $T_1 \in \mathcal{A}$.

<u>Sketch</u> of the proof of theorem 3. We extend the ideal $(\mathcal{A}, \alpha)$ in L(H) to an injective and surjective quasinormed ideal in the class of all Banach spaces. Let $T \in \mathcal{A}^{\mathcal{L}}$. Its extension T_o according to theorem 1 satisfies the assumption of theorem 3. Therefore, T_o has a factorization through $T_1 \in \mathcal{A}$. Using proposition 2 we can replace B_2 and B_3 by the Hilbert space H. This yields a factorization

$$\begin{array}{ccc} D' & \xrightarrow{\quad T_o \quad} & D \\ {\scriptstyle Q'}\downarrow & & \uparrow{\scriptstyle P} \\ H & \xrightarrow{\quad T_1 \quad} & H \end{array}$$

with $Q, P \in \mathcal{B}_1$. Restricting T_o to D we obtain $T_o | D = T = PT_1Q^*$.

3. <u>Final remarks</u>

Dual to the definition of $\mathcal{A}^{\mathcal{L}}$ you can define the set

$$\mathcal{A}^{\mathcal{B}} = \{ S \in L^+(D) : \mathcal{B}_1^* \cdot S \cdot \mathcal{B}_1 \subsetneqq \mathcal{A} \}.$$

This set is also an $*$ - ideal in $L^+(D)$, in generally it contains unbounded operator. There is the following theorem concerning the structure of $\mathcal{A}^{\mathcal{B}}$. An ideal $\mathcal{A}$ is called perfect, if it coincides with its second adjoint ($\mathcal{A} = \mathcal{A}^{**}$, see /4/).

<u>Theorem 4</u>. If $(\mathcal{A}, \alpha)$ is a perfect ideal in L(H), then

$$\mathcal{A}^{\mathcal{B}} = L^+(D)' \cdot \mathcal{A} \cdot L^+(D) ,$$

where $L^+(D)' = \{ X' : D' \longrightarrow D' \mid X \in L^+(D)$.

Now we return to the problem of the trace representation of the linear functionals on ideals.

<u>Proposition 3</u>. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two normed complete ideals in L(H) such that their product $\mathcal{A}_1 \cdot \mathcal{A}_2$ contains only nuclear operators. Then a dual pair $\langle \mathcal{A}_1^{\mathcal{L}}, \mathcal{A}_2^{\mathcal{B}} \rangle$ is defined by

$$\langle T, S \rangle = \langle PT_1Q^*, S \rangle := \text{trace } Q^*SP \cdot T_1 .$$

Using this dual pair it is now possible to define dual pair topologies on $\mathcal{A}_1^{\mathcal{L}}$ and $\mathcal{A}_2^{B}$. Then the linear continuous functionals are exactly the trace functionals.

References.

/1/ Junek, H.: Operatorenideale und lokalkonvexe Räume. Diss. (B), Potsdam 1979.

/2/ Junek, H./Müller, J.: Topologische Ideale unbeschränkter Operatoren im Hilbertraum, Wiss. Ztschr. PHKL Potsdam 25 (1981) 101-110.

/3/ Laßner, G.: Topological algebras of operators. Rep. Math. Phys. 3 (1972) 279-293.

/4/ Pietsch, A.: Operator ideals, Berlin 1978.

/5/ Timmermann, W.: Ideals in algebras of unbounded operators, Prepr. Dubna E5-10758 (1977).