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LOCAL UNIFORM CONVEXITY OF DAY'S NORM ON $c_0(\Gamma)$

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The object of this note is to give an alternate proof of the famous theorem of Rainwater [2] that Day's norm [1] on $c_0(\Gamma)$ is locally uniformly convex. The main feature of our proof is that it does not rely on auxiliary results involving sequences and permutations as, for example, (2) p. 336 of [2]. Further, our proof has the merit of being easier for presentation in a course.

1. Let Γ be a set. The space $c_0(\Gamma)$ is the Banach space of all real valued continuous functions x on Γ such that $\{\gamma\in\Gamma: |x(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$, with the supremum norm. M.M. Day's norm [1] on $c_0(\Gamma)$ can be expressed as follows: Let Φ be the set of all sequences $\phi = \{\gamma_n\}$ in Γ. Define $F_\phi: c_0(\Gamma) \to \ell_2$ by $[F_\phi x](n) = 2^{-n}x(\phi(n))$. Then Day's norm is

$$(1) \qquad ||x|| = \sup\{||F_\phi x||_{\ell_2}: \phi\in\Phi\} .$$

The supremum is attained for any ϕ for which the sequence $x(\phi(n))$ is non-increasing. Thus, if $E(x) = \{\gamma_n\}$ is the support of x enumerated so that $|x(\gamma_k)| \geq |x(\gamma_{k+1})|$ for all k, then

$$||x|| = \{\Sigma_k 4^{-k} x(\gamma_k)^2\}^{1/2} .$$

Day proved that the function $||\ ||$ is actually a norm on $c_0(\Gamma)$ and

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that it is strictly convex (rotund). Further $\frac{1}{2}||x||_{c_0(\Gamma)} \leq ||x|| \leq ||x||_{c_0(\Gamma)}$.

2. Theorem (Rainwater). Day's norm on $c_0(\Gamma)$ is locally uniformly convex, i.e., given x and a sequence $\{x_n\}$ in $c_0(\Gamma)$ such that $||x|| = 1$, $||x_n|| = 1$, $n = 1,2,\ldots,$ and $||x+x_n|| \to 2$, then $||x-x_n|| \to 0$.

Proof: We shall show that for each $\{\gamma_n\}$ in Γ it is true that $(x-x_n)(\gamma_n) \to 0$.

Without loss of generality we may assume that the sequences $\{x(\gamma_n)\}$ and $\{(x+x_n)(\gamma_n)\}$ are convergent, that $(x+x_n)(\gamma_n) \neq 0$ for all n and that one of the following cases hold:

(A) $\gamma_n = \gamma \quad n = 1,2,\ldots$

(B) γ_n's are all different.

Let $E(x) = \{\alpha_k\}$, $E(x_n) = \{\alpha_k^n\}$ and $E(x+x_n) = \{\beta_k^n\}$ be the supports of x , x_n and $x+x_n$ respectively, enumerated so that, for $n,k = 1,2,\ldots,$

$$|x(\alpha_k)| \geq |x(\alpha_{k+1})| \text{ , } |x(\alpha_k^n)| \geq |x(\alpha_{k+1}^n)| \text{ and} \tag{2}$$

$$|(x+x_n)(\beta_k^n)| \geq |(x+x_n)(\beta_{k+1}^n)| \text{ .}$$

Since, for each k , the sequence $\{x(\beta_k^n)\}$ is bounded we may choose se uences $\{n_i^k\}_{i=1}^{\infty}$, $k = 1,2,\ldots,$ such that $\{n_i^k\} \supset \{n_i^{k+1}\}$ and

$\{x(\beta_k^{n_i^k})\}_{i=1}^{\infty}$ is convergent. It follows that $\{x(\beta_k^{n_i^i})\}_{i=1}^{\infty}$ is convergent, for each k, say, to b_k. From now on, we shall be considering only the subsequence $\{n_i^i\}_{i=1}^{\infty}$ and so, for simplicity, we drop the i's and write

$$\lim_{n\to\infty} x(\beta_k^n) = b_k \text{ , } k = 1,2,\ldots \tag{3}$$

It follows from (1) that

$$||x||^2 = \sum_k 4^{-k}x(\alpha_k)^2 \geq \sum_k 4^{-k}x(\beta_k^n)^2 \text{ .}$$

Using this and similar inequalities for x_n, $n = 1,2,\ldots$, we get

$$\begin{aligned} 4-||x+x_n||^2 &= 2||x||^2 + 2||x_n||^2 - ||x+x_n||^2 \\ &= \sum_k 4^{-k}[2x(\alpha_k)^2+2x_n(\alpha_k^n)^2-(x+x_n)(\beta_k^n)^2] \\ &\geq \sum_k 4^{-k}[2x(\beta_k^n)^2+2x_n(\beta_k^n)^2-(x+x_n)(\beta_k^n)^2] \\ &= \sum_k 4^{-k}[x(\beta_k^n)-x_n(\beta_k^n)]^2 \text{ .} \end{aligned} \tag{4}$$

By assumption $||x+x_n||^2 \to 4$ and, so using (3), we obtain, for each k,

$$\lim_{n\to\infty} x_n(\beta_k^n) = \lim_{n\to\infty} x(\beta_k^n) = b_k \tag{5}$$

and further,

$$\lim_{n\to\infty}(x+x_n)(\beta_k^n) = 2b_k \text{ .} \tag{6}$$

Then, from the last inequality of (2) we get

$$b_1^2 \geq b_2^2 \geq \dots \geq b_k^2 \geq \dots . \tag{7}$$

Since

$$\begin{aligned} \sum_k 4^{-k} b_k^2 &= 4^{-1} \lim_{n\to\infty} \sum_k 4^{-k} (x+x_n)(\beta_k^n)^2 \\ &= 4^{-1} \lim_{n\to\infty} \|x+x_n\|^2 = 1 \end{aligned} \tag{8}$$

we must have at least $b_1 \neq 0$, and so, by virtue of (5) the sequence $\{x(\beta_1^n)\}$ is constant for large n's. Thus there must exist β_1 such that $\beta_1 = \beta_1^n$ for infinitely many n, say for all i in a sequence $\{n_i^1\}$. It is obvious that $\beta_1 = \alpha_{i_1}$ for some $\alpha_{i_1} \in E(x)$.

Suppose we have already sequences

$$\{n_i^1\} \supset \{n_i^2\} \supset \dots \supset \{n_i^m\}$$

and different points $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m}$ such that

$$\beta_k^{n_i^k} = \alpha_{i_k}, \quad k=1,\dots,m \quad \text{and} \quad i=1,2,\dots,$$

and $b_m \neq 0$. If $b_{m+1} \neq 0$, then we apply the preceding method to get $\{n_i^{m+1}\} \subset \{n_i^m\}$ and $\alpha_{i_{m+1}} \in E(x)$ such that $\beta_{m+1}^{n_i^{m+1}} = \alpha_{i_{m+1}}$ for all i. Clearly we have, for all $k = 1,2,\dots,m+1$, the equality

$$\alpha_{i_k} = \beta_k^{n_1^{m+1}}$$

and, by the definition of $E(x+x_n)$, all members of $\{\beta_k^n\}$, k=1,2,... are distinct, and so, if $j \neq k$, $1 \leq j < k < m+1$, then $\alpha_{i_j} \neq \alpha_{i_k}$.
If there exists m such that $b_m \neq 0$ but $b_{m+1} = 0$, then we denote the sequence n_i^m by $\{n_i\}$, and if all b_k are non-zero we denote by $\{n_i\}$ the sequence $\{n_i^i\}$. It follows, then, that $\{n_i\}$ has the following property: for each k with $b_k \neq 0$ we have $\beta_k^{n_i} = \alpha_{i_k}$ and consequently $b_k = x(\alpha_{i_k})$ for all sufficiently large i. Then, by (8), we have

$$\sum_k 4^{-k} x(\alpha_{i_k})^2 = 1$$

and since all the points α_{i_k} are different, we see that $\{\alpha_{i_k}\}$ is only a permutation of $\{\alpha_k\}$. So, without loss of generality, we may enumerate $E(x)$ so as to have $\alpha_{i_k} = \alpha_k$ and rewrite (5) in the form

$$\lim_{n\to\infty} x_n(\alpha_k) = x(\alpha_k) = b_k \text{, for all } k . \tag{9}$$

In particular, we have $b_k \to 0$. Using this, (6) and the last inequality of (2), we see that, for every infinite sequence $\{k_n\}$ and any increasing sequence $\{n_i\}$

$$\lim_{n\to\infty}(x+x_{n_i})(\beta_k^{n_i}) = 0 . \tag{10}$$

We claim now that there is a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that

(11) $$\lim_{n\to\infty}(x-x_n)(\gamma_{n_j}) = 0 \ .$$

To see this, suppose (A) holds. If $\gamma \in E(x)$, then (11) follows from (9). If $\gamma \notin E(x)$, then, by assumption, we have $\gamma \in E(x+x_n)$, i.e., $\gamma = \beta^n_{k_n}$, n=1,2,... If $\{k_n\}$ is infinite, (11) follows from (10) and if there exists k_0 such that $\gamma = \beta^n_{k_0}$ for infinitely many n, we deduce (11) from (5).

On the other hand, suppose (B) holds. Then, since $x \in c_0(\Gamma)$ we have $x(\gamma_n) \to 0$. If there is $\{n_k\}$ such that $\gamma_{n_k} \notin E(x+x_{n_k})$, then, we have also $x_{n_k}(\gamma_{n_k}) \to 0$ and so $(x-x_{n_k})(\gamma_{n_k}) \to 0$ and (11) is true. If $\gamma_n \in E(x+x_n)$ for all sufficiently large n, then $\gamma_n = \beta^n_{k_n}$. Then assumption (B) implies that $\{k_1, k_2, \ldots\}$ is an infinite set and (11) follows from (10). This completes the proof.

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