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NINTH WINTER SCHOOL ON ABSTRACT ANALYSIS (1981)

Feynman path integral as spectral decomposition

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In previous lecture we introduced the Fock-Stueckelberg space $\mathcal{H}$ and the propagator $\mathcal{K}$ on it. Now we shall try to give another representation of the operator $\mathcal{K}$ which does not depend on the splitting of $\mathcal{K}$ in $\mathcal{K}_o$ and $\mathcal{J}$ but is completely determined by classisal action (or by Lagrangian). We also hint to the connection of this formulation of QFT with Feynman path integral.

For simplicity of our considerations we shall assume only the theory of one scalar field, i.e. the Lagrangian

$$\mathcal{L}(\varphi,x) = \frac{1}{2}\left[(\partial_\mu\varphi(x))^2 - m^2\varphi^2(x)\right] - \frac{\lambda}{4}\varphi^4(x) - J(x)\varphi(x), \qquad (1)$$

we also take into account the external source $J(x)$. Quantization in our formalism is a construction of operator $\mathcal{K}$ on space $\mathcal{H}$. The construction of this operator from classical Lagrangian we shall call the 'kinematical quantization'. The procedure performs as follows

a/ to the classical field $\varphi(x)$ assign the operator distribution ϕ_x on $\mathcal{H}$ which has the expression in creation and anihilation operators from previous lecture

$$\varphi(x) \longrightarrow \phi_x = \frac{1}{\mu_o}\left(a_x^+ + a_x\right) = \phi_x^+ + \phi_x^-$$

with μ_o being a massive parameter appeared only due to dimensional reasoning (i.e. the action is dimensionless). That will be finally removed from the theory. Its interpretation will be clear later on.

b/ construct the operator $\mathcal{K}$ from quantum action in the manner

$$\mathcal{K} = \exp\left\{ i\, \mathcal{A}_Q \right\} = \exp\left\{ i \int d^4x\, \mathcal{L}(\phi_x) \right\} .$$

We emphesize here that ϕ_x is a distributive operator and derivatives act on ϕ_x in similar way as on number-valued distributions. Due to distributive character of ϕ_x quantum action has not proper mathematical meaning without renormalization or without regularizing the theory. We do not want to treat these problems here.

From the previous lecture we know the interpretation of matrix elements of $\mathcal{K}$ - they are just Green's functions G

$$G_{\mu_0}(x_1, \dots x_n) = \langle 0 | \phi_{x_1} \dots \phi_{x_n} \mathcal{K} | 0 \rangle \quad , \qquad (2)$$

we added the index μ_0 to a matrix element to express explicitly the dependence on parameter μ_0 . We prove

$$G(x_1, \dots x_n) = \lim_{\mu_0 \to 0} G_{\mu_0}(x_1, \dots x_n) \qquad (3)$$

to be valid nonperturbatively from the equations of motion, i.e. equations for Green's functions. Knowing commutation relations and expression for $\mathcal{K}$ in ϕ_x we derive the quantum equations of motion for operator $\mathcal{K}$. From following commutation relations

$$[\phi_x^-, \phi_y^+] = \frac{1}{\mu_0^2}\, \delta(x-y) \quad , \quad [\phi_x, \phi_y] = 0 \qquad (4)$$

we obtain $\phi_x \mathcal{K} = \phi_x^+ \mathcal{K} + \mathcal{K} \phi_x^- + i [\phi_x^-, \mathcal{A}]$

From definition of $\mathcal{A}$ we have

$$[\phi_x^-, \mathcal{A}] = -\frac{1}{\mu_0^2} \left[(\partial^2 + m^2)\phi_x + \lambda \phi_x^3 + J(x) \right] \text{ , i.e.}$$

$$\left[(\partial^2 - i\mu_0^2 + m^2)\phi_x + \lambda\phi_x^3 + J(x) \right] \mathcal{K} = -i\mu_0^2 \left(\phi_x^+ \mathcal{K} + \mathcal{K}\phi_x^- \right) \qquad (5)$$

which are equations of motion for $\mathcal{K}$ that is understood to be a functional of Φ_x and J(x). These operator equations are compactly represented equations for Green's functions. From (2) and (5) follows

$$\left(\partial_x^2 - i\mu_o^2 + m^2\right) G_{\mu_o}(x,x_1,\dots x_n) + \lambda G_{\mu_o}(x,x,x,x_1,\dots x_n) +$$
$$+ J(x) G_{\mu_o}(x_1,\dots x_n) = -i\sum_{k=1}^{n} \delta(x-x_k) G_{\mu_o}(x_1,\dots x_{k-1},x_{k+1},\dots x_n), \qquad (6)$$

having in mind the limiting process $\mu_o \to 0$, (6) are the equations for Green's functions. (6) make clear the interpretation of μ_o being the Feynman causal ε.

To restore the generating functional from $\mathcal{H}$ we use spectral decomposition in a special basis (Feynman basis). Since $[\phi_x, \phi_y] = 0$ and we suppose ϕ_x to be an irreducible ring of operators, we can choose the basis that diagonalizes ϕ_x simultaneously, i.e. $\phi_x |\varphi\rangle = \varphi(x)|\varphi\rangle$, $\varphi(x) \in \mathcal{S}'(\mathbb{R}^4)$, where $\varphi(x)$ is called classical configuration. Such basis is not from $\mathcal{H}$, as ϕ_x are distributive operators, so $|\varphi\rangle$ lie in some suitable extension of $\mathcal{H}$. In Feynman basis we can formally introduce the projector measure

$$[d\varphi]\, |\varphi\rangle\langle\varphi| = |\varphi\rangle\langle\varphi| \prod_{x\in\mathbb{R}^4} d\,\varphi(x) ,$$

in this measure the decomposition of unity is

$$\mathbb{1} = \int |\varphi\rangle\langle\varphi|\, [d\varphi] \quad ,$$

ortogonality $\langle\varphi|\varphi'\rangle = \delta(\varphi|\varphi')$, $\int[d\varphi']\, \Psi(\varphi')\, \delta(\varphi|\varphi') = \Psi(\varphi)$

A vector from $\mathcal{H}$ we can represent by the vector measure

$$|\Psi\rangle = \int [d\varphi]\, \Psi(\varphi)\, |\varphi\rangle \quad , \qquad \Psi(\varphi) = \langle\varphi|\Psi\rangle \; .$$

Operator $\mathcal{K}$ has the spectral decomposition

$$\mathcal{K} = \int e^{iA(\varphi,J)} |\varphi\rangle\langle\varphi| \, [d\varphi]$$

and the generating functional Z (we let $\lambda = 0$ in (1)) is

$$Z(J) = \langle 0|\mathcal{K}|0\rangle = \int e^{iA(\varphi,J)} |\langle 0|\varphi\rangle|^2 [d\varphi] = \int e^{iA(\varphi,J)} d\mu(\varphi),$$

where $d\mu(\varphi)$ is Feynman measure. To obtain it we use the definition of virtual momentum π_x: $[\pi_x, \phi_y] = -i\,\delta(x-y)$. (7)

From (4) and (7) we get

$$\pi_x = \frac{-i\mu_o^2}{2}(\phi_x^- - \phi_x^+), \quad \phi_x^- = \frac{1}{2}\left(\frac{2i}{\mu_o^2}\pi_x + \phi_x\right). \qquad (8)$$

Vacuum is characterized $\phi_x^- |0\rangle = 0$. (9)

Using the representation of a state in Feynman basis we have

$$|0\rangle = \int [d\varphi]\, \Psi_o(\varphi) |\varphi\rangle \quad \text{and} \quad \phi_x |0\rangle = \int [d\varphi]\, \varphi(x)\, \Psi_o(\varphi) |\varphi\rangle ,$$

$\pi_x |0\rangle = \int [d\varphi]\left(\frac{-i\,\delta}{\delta\varphi(x)} \Psi_o(\varphi)\right)|\varphi\rangle$. From (8) and (9) we have in Feynman basis the functional equation

$$\varphi(x) + \frac{2}{\mu_o^2} \frac{\delta}{\delta\varphi(x)} \Psi_o(\varphi) = 0$$

with solution $\Psi_o(\varphi) = \exp\left\{-\frac{\mu_o^2}{4}\int \varphi^2(x)\, d^4x\right\}$. Thus generating functional is expressed

$$Z(J) = \int e^{iA(\varphi,J)} \exp\left\{-\frac{\mu_o^2}{2}\int \varphi^2(x)\, d^4x\right\}[d\varphi]$$

which is just the Feynman's path integral representation for generating functional /1/ with $\mu_o \to 0$. Green's functions are obtained by functional derivating that gives the functions defined by (2) and (3).

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Manifestly covariant equivalent to quantum field theory

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In 1942 E.C. Stueckelberg /1/ made an attempt to formulate quantum mechanics in a covariant space of wawe functions ψ, i.e. he used the four-interval normalization of $\psi(x)$, $x \in M$ (M is Minkowski space): $\int |\psi(x)|^2 d^4x = 1$. The space spanned on such functions forms $L_2(M)$ and we call it the Stueckelberg space. In 1949 R.P. Feynman /2/ developped the method of diagrams which is up to now the basic ingredient of the quantum field theory (QFT). Feynman started with the formula

$$K(1|2) = K_0(1|2) + ie\int d^4 3\ K_0(1|3)V(3)K_0(3|2) + \dots \qquad (1)$$

where $K(1|2)$ is the total propagator of a particle from 1 to 2 ($1,2 \in M$), $K_0(1|2)$ is a free propagator of a particle, $V(3)$ is an interaction. We now shall try to generalize (1) for many-partical case, i.e. $K, K_0 \to \tilde{\mathcal{K}}, \tilde{\mathcal{K}}_0$ operators on Fock-Stueckelberg space. This generalization looks like

$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_0 + \frac{ie}{1!}\tilde{\mathcal{K}}_0\tilde{\mathcal{J}}\tilde{\mathcal{K}}_0 + \frac{(ie)^2}{2!}\tilde{\mathcal{K}}_0\tilde{\mathcal{J}}\tilde{\mathcal{K}}_0\tilde{\mathcal{J}}\tilde{\mathcal{K}}_0 + \dots = e^{\tilde{\mathcal{K}}_0\tilde{\mathcal{J}}}\tilde{\mathcal{K}}_0 \qquad (2)$$

and has the same Feynman interpretation for many particles as (1) for one particle.

The Fock-Stueckelberg space is constructed from Stueckelberg spaces $H_S = L_2(M)$:

$$\mathcal{H}_{FS} = \sum_{n=0}^{\infty} \oplus H_n , \qquad H_n = \begin{cases} \bigodot_{k=1}^{n} H_S & \text{for bosons} \\ \bigwedge_{k=1}^{n} H_S & \text{for fermions} \end{cases}$$

$\odot$, $\wedge$ are symetrized and antisymetrized tensor products.

H_o is zero particle state with scalars ($\mathcal{C}$ numbers). As usually we introduce creation and anihilation operators (operator distributions) : a_x, a_x^+ with commutation relations $[a_x, a_y^+] = \delta^{(4)}(x-y)$ $x, y \in M$. If $|0\rangle$ is from H_o with $a_x|0\rangle = 0$, then $\mathcal{H}_{FS}$ is spanned on $|0\rangle$, $a_{x_1}^+$, ..., $a_{x_n}^+|0\rangle$, $n = 1,2, \ldots$. We shall use the suitable multipoint notation :

$$M^{(n)} = \{x = (x_1, \ldots, x_n) | x_1, \ldots, x_n \in M\},$$

$$M^F = \bigcup_{n=0}^{\infty} M^{(n)} \quad , \quad M^F \text{ is Fock-Minkovski space},$$

$$a_x = (n!)^{-\frac{1}{2}} \cdot a_{x_n} \ldots a_{x_1} \quad , \quad x = (x_1, \ldots, x_n)\ M^F$$

and we shall also use the Einstein's summation rule - summing (integrating) over all double repeated (not appeared in parenthesis) indexes. The completeness relations for $\mathcal{H}_{FS}$ can be put down as

$$1\!\!1 = a_y^+|0\rangle\langle 0|a_x^- \ \delta(y|x) = a_x^+|0\rangle\langle 0|a_x^- \quad , \tag{3}$$

with $\delta(y|x) = \langle 0|a_y a_x^+|0\rangle$, $x, y\ M^F$.

The many particle wawe function ψ on $\mathcal{H}_{FS}$ will be $\psi(x) = \langle x|\psi\rangle$, $x \in M^F$ or $|\psi\rangle = \psi(x)|x\rangle$, $|x\rangle = a_x^+|0\rangle$. We must emphesize here that $\psi(x)$ has interpretation as multiparticle amplitude for states localized in space-time points /3/, i.e. it describes the virtual states of the physical particles. Among these virtual states exist also the physical states, that means - physical states form a subspace $\mathcal{H}_{phys}$ of $\mathcal{H}_{FS}$. $\mathcal{H}_{phys}$ is spanned on states $|\vec{p}, \omega(\vec{p})\rangle = a^+_{\vec{p},\omega}|0\rangle = e^{-ipx}\ a_x^+|0\rangle$, $p_o = \omega(\vec{p})$, $\omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$, $\vec{p} \in R^3$, i.e. on states with correct dispersion relation for four-momentum. Now we are able to define the operator $\mathcal{K}_o$. We want

$\langle y|\mathcal{K}_0|x\rangle$, $x,y\in \mathbb{M}^F$ to be the transition amplitude of many-particle state $|x\rangle$ to many-particle state $|y\rangle$. When $\mathcal{K}_0$ is a free propagator, then it must be diagonal (preserving the particle number), so we write

$$\mathcal{K}_0 = a_y^+|y\rangle\langle x|a_x\mathcal{K}_0(y|x)\ ,\ x,y\in \mathbb{M}^F\ ,$$

$$\mathcal{K}_0(y|x) = \delta_{|y|,|x|}K_0(y_1|x_1)\dots K_0(y_{|x|}|x_{|x|}),$$

where $|x|$ is a length of a multipoint x (number of components), $K_0(y_1|x_1)$ is a complex function. To define the full propagator $\widetilde{\mathcal{K}}$ we are to define an operator of interaction $\widetilde{\mathcal{J}}$. It will certainly be that changing the number of particles in the process. The general form of $\widetilde{\mathcal{J}}$ is $\widetilde{\mathcal{J}} = a_y^+ J(y|x)a_x$,$x,y\in \mathbb{M}^F$, where $J(y|x)$ is some integral kernel (may be nonlocal). To restore the formula (2) i.e. the full propagator is a sum of succesive iterations - particles are propagated freely, then some of them interact and interacted particles again are freely propagated; we must introduce a new isomorfic copy $\widetilde{\mathcal{H}}_{FS}$ of $\mathcal{H}_{FS}$ and their tensor product $\mathcal{H} = \mathcal{H}_{FS}\otimes\widetilde{\mathcal{H}}_{FS}$. We shall denote $\tilde{a}_x$, $\tilde{a}_x^+$ anihilation and creation operators in $\widetilde{\mathcal{H}}_{FS}$ and identify $a_x^+\otimes\tilde{1}\to a_x^+$ $1\otimes\tilde{a}_x^+\to\tilde{a}_x^+$, $|0\rangle\otimes|\tilde{0}\rangle\to|0\rangle$ whenever confusion is excluded.

The operator $\mathcal{K}_0$ will be now the operator $\widetilde{\mathcal{K}}_0$

$$\widetilde{\mathcal{K}}_0 = a_y^+ P_0\ \tilde{a}_x \mathcal{K}_0(y|x)\ ,\ P_0 = 1\otimes|\tilde{0}\rangle\langle\tilde{0}|$$

that transforms particles from $\widetilde{\mathcal{H}}_{FS}$ (in-space) into $\mathcal{H}_{FS}$ (out-space) leaving $\mathcal{H}_{FS}$ without change. Interaction operator is then modified in the way $\widetilde{\mathcal{J}} = \tilde{a}_y^+ J(y|x)a_x$. From $\widetilde{\mathcal{K}}_0$ and $\widetilde{\mathcal{J}}$ we construct the formula (2) having in mind the in-states are from $\widetilde{\mathcal{H}}_{FS}$, out-states are from $\mathcal{H}_{FS}$. Matrix element of operator $\widetilde{\mathcal{K}}$, its n-th iteration is given

$$(n!)M_{fi} = \langle 0| a_f \tilde{\mathcal{K}}_0 \tilde{\mathcal{J}} \tilde{\mathcal{K}}_0 \dots \tilde{\mathcal{J}} \tilde{\mathcal{K}}_0 \tilde{a}_i^+ |0\rangle = \langle 0| a_f \dots a^+_{y_k} \mathcal{K}_0(y_k|x_k) \times$$
$$\times \mathbb{P}_0 \tilde{a}_{x_k} \tilde{a}^+_{u_k} P_0 \, J(u_k|v_k) a_{v_k} \dots \tilde{a}_i^+|0\rangle = \langle 0| a_f \dots a^+_{y_k} \mathcal{K}_0(y_k|x_k) J(x_k|v_k)$$
$$a_{v_k} \dots a^+_{x_n} \mathcal{K}_0(x_n|i)|0\rangle ,$$

where we used the definition of the projector P_0. We define a dressed creation operator $a_x^{(+)} = a_y^+ \mathcal{K}_0(y|x)$ and a left dressed interaction operator $\mathcal{J}^{(dr)} = a_y^{(+)} J(y|x)\, a_x$. Then
$(n!)M_{fi} = \langle 0| a_f \,\mathcal{J}^{(dr)} \dots \mathcal{J}^{(dr)} a_i^{(+)}|0\rangle$, hence $\mathcal{K} = e^{\mathcal{J}^{(dr)}} \mathcal{K}_0$,

when out and in states are from $\mathcal{H}_{FS}$ (in-states are not dressed). That means, if we are interested in matrix elements of $\mathcal{K}$ only (i.e. in transition amplitudes) the space $\mathcal{H}_{FS}$ is sufficient.

Now we show what K_0 and J give the results of canonical QFT; give the same Green's functions.

1) Neutral scalar field.

We choose $K_0(x|y) = iD_F(x-y)$, D_F is Feynman causal propagator, $\tilde{\mathcal{J}} = \lambda \int d^4x (a_x + \tilde{a}^+_x)^4$. This leads to
$\mathcal{J}^{(dr)} = \lambda \int d^4x \, {:}\phi_x^4{:}$, $\phi_x = (a_x + a_x^{(+)})$, $:\;:$ are Wick doubledots.
We shall denote canonical QFT quantities by $\hat{}$, so $\hat{\phi}_x$ will be the quantum field in interaction representation. The Green's function in n-th order of perturbation series is

$$(n!)\hat{M}_{fi} = \lambda^n \langle 0| T[\hat{\phi}_{f_1} \dots \hat{\phi}_{f_{|f|}} {:}\hat{\phi}^4_1{:} \dots {:}\hat{\phi}^4_n{:}\, \hat{\phi}_{i_1} \dots \hat{\phi}_{i_{|i|}} |0\rangle .$$

This expression equals to all chronological contractions (Wick's theorem). The element M_{fi} in our theory

$$(n!)M_{fi} = \langle 0| \phi_{f_1} \dots \; {:}\phi^4_1{:} \; \dots \; {:}\phi^4_n{:} \dots \phi_{i_1}|0\rangle \lambda^n$$

is equal to the sum of all normal contractions. But

$$\overbrace{\phi_1 \phi_2} = \langle 0| \phi_1 \phi_2 |0\rangle = \langle 0| a_1 a_2^+ |0\rangle = iD_F(1-2) = \langle 0| T[\phi_1 \phi_2]|0\rangle =$$

$= \overline{\hat{\phi}_1 \hat{\phi}_2}$ and we have $\hat{M}_{fi} = M_{fi}$. So $\langle 0|\phi_f\, e^{\mathcal{J}^{(dr)}}\phi_i|0\rangle = \langle 0|\, T[\hat{\phi}_f \hat{\phi}_i \hat{S}]|0\rangle$.

2/ Q.E.D

Several changes must be done in this theory. Firstly, we must introduce a new space $\mathcal{H}$ for antiparticles with operators b_x, b_x^+. Now, fields a_x, b_x are fermionic, the anticommutators must be considered. All formulas remain valid, only $\delta(y|x)$ in (3) has different dependence on arguments – the signum of permutation must be considered. Thus we have three particles – electrons and positrons as fermions and photons as bosons with /anti/ commutation relations

$$[a_x, a_y^+]_+ = [b_x, b_y^+]_+ = \delta^{(4)}(x-y),\quad [c_{x\mu}, c_{y\nu}^+] = g_{\mu\nu}\delta^{(4)}(x-y)\ .$$

The choice for free propagators and interaction is

$$K_0(x|y) = iS_F(x-y),\ \overline{K}_0(x|y) = -iS_F^T(y-x),\ D_{0\mu\nu}(x|y) = iD_{F\mu\nu}(x-y),$$

where T is transposition of spinor indexes which are suppressed. K_0 propagates electrons, $\overline{K}_0$ positrons, D_0 photons.

$\tilde{\mathcal{J}} = ie\int d^4x\ (\,b_x + \tilde{a}_x^+)\,\gamma^\mu(a_x + \tilde{b}_x^+)\,(\tilde{c}^+_{x\mu} + c_{x\mu})$ giving

$\mathcal{J}^{(dr)} = ie\int d^4x\ :\overline{\Psi}_x\, \gamma^\mu \Psi_x\, A_{x\mu}:$, with $\Psi_x = (a_x + b_x^{(+)})$,

$\overline{\Psi}_x = (b_x + a_x^{(+)})$, $A_{x\mu} = (c_{x\mu} + c_{x\mu}^{(+)})$. The same arguments show the equality of Green's functions in both theories.

At the end of this lecture we show formula (2) reduces to (1) in the case of one particle in external field. Starting with $\mathcal{J}^{(dr)} = i\lambda\int d^4x\, \mathcal{V}(x)\ :(a_x + b_x^{(+)})(b_x + a_x^{(+)}):$, denoting $\mathcal{J}^{(IS)} = i\lambda\int d^4x\, \mathcal{V}(x)\, a_x^+ a_x$ we have

$$\frac{\langle 0|\, a_f e^{\mathcal{J}^{(dr)}}\, a_i^{(+)}|0\rangle}{\langle 0|\, e^{\mathcal{J}^{(dr)}}|0\rangle} = \sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{\langle 0|\, a_f[\mathcal{J}^{dr}]^n\, a_i^{(+)}|0\rangle}{\langle 0|\, e^{\mathcal{J}^{(dr)}}|0\rangle} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\binom{n}{k}\ \times$$

$$\times \frac{\langle 0|a_f[\mathcal{Y}^{(dr)}]^n\, a_i^{(+)}|0\rangle_c \langle 0|[\mathcal{Y}^{(dr)}]^{n-k}|0\rangle}{\langle 0|e^{\mathcal{Y}^{(dr)}}|0\rangle} = \sum_{k=0}^{\infty} \langle 0|a_f[\mathcal{X}_0 \mathcal{Y}^{(LS)}]^k a_i^{(+)}|0\rangle =$$

$$= \langle 0|a_f[1-\mathcal{X}_0\mathcal{Y}^{(LS)}]^{-1}\mathcal{X}_0\, a_i^{+}|0\rangle \quad ,$$

where $\langle\ \rangle_c$ denotes connected graphs. This proves the correct reduction of (2) to (1) in this case.

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