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RAPID ULTRAFILTER NEED NOT BE Q-POINT

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Clasifying points in $\beta\omega$ mathematicians have introduced several types of ultrafilters on the set ω of natural numbers. Some of them are interesting also from other point of view. In this note we shall consider two important types of ultrafilters: rapid and Q-point. Both notions are applicable to filters too.

The existence of rapid ultrafilters and Q-points is undecidable in the set theory. The continuum hypothesis implies the existence of Q-points. Every Q-point is a rapid filter [1]. There exists a model of ZFC in which there is no rapid ultrafilter [5]. Of course, there exists an ultrafilter which is not rapid [1]. This is proved by observing that every rapid filter $\mathcal{U}$ has the following property (C): if $\{a_n\}_{n=0}^{\infty}$ is a sequence of positive reals converging to zero then there exists a set $A \in \mathcal{U}$ such that $\sum_{n \in A} a_n < +\infty$.

This property is characteristic for rapid filters.

<u>Proposition</u>. If a filter $\mathcal{U}$ possesses the property (C) then $\mathcal{U}$ is rapid.

Probably this proposition is known. We did not find it in literature, therefore we present a simple proof of it.

The main goal of this note is the following

<u>Theorem A</u>. Let M be a transitive model of ZFC. Then there exists a generic extension N of M such that

a) cardinals of N are those of M;

b) $(2^{\aleph_0})^M = (2^{\aleph_0})^N$;

c) in N there exists a rapid filter $\mathcal{V}$ such that $\mathcal{V}$ is not a Q-point.

Moreover we can assume that $\mathcal{V}$ is a P-point (and ultrafilter).

By slight modifications of the forcing construction used to prove the theorem A we shall obtain a proof of

<u>Theorem B</u>. Assume that there exists a dominating family $\mathcal{F} \subseteq {}^{\omega}\omega$

of cardinality λ . If MA_{κ} holds true for every $\kappa < \lambda$ then there exists a rapid filter $\mathcal{V}$ such that $\mathcal{V}$ is not a Q-point. Moreover we can assume that $\mathcal{V}$ is an ultrafilter.

§1. Preliminaries. If A is an infinite set of integers then $\bar{A}$ is the counting function of A, i.e. $\bar{A}$ is the unique strictly increasing function from ω onto A. A family of functions $\mathcal{F} \subseteq {}^{\omega}\omega$ is dominating iff for every $f \in {}^{\omega}\omega$ there exists a function $g \in \mathcal{F}$ and a $k \in \omega$ such that $g(n) \geq f(n)$ for every $n \geq k$. A filter $\mathcal{U}$ on ω is rapid iff the family $\{\bar{A};\ A \in \mathcal{U}\}$ is dominating. Evidently, any extension of a rapid filter is rapid. A filter $\mathcal{U}$ is a Q-point iff for any partition $\mathcal{A} = \{A_n; n \in \omega\}$ of ω ,A_n being finite, there exists a set $A \in \mathcal{U}$ such that $|A \cap A_n| \leq 1$ for every $n \in \omega$. The set A is called a selector for the partition $\mathcal{A}$.

For the rest of the paper we fix a partition $\mathcal{R} = \{R_n; n \in \omega\}$ of ω such that $|R_n| = n$. E.g. we set $R_n = \{\frac{1}{2}n(n-1), \dots, \frac{1}{2}n(n-1) + n-1\}$. A set $A \subseteq \omega$ will be called growing iff for every $n \in \omega$ there exists a $k \in \omega$ such that $|A \cap R_k| \geq n$. We denote

$$\mathcal{S} = \{A \subseteq \omega;\ \exists n \forall k |R_k - A| \leq n\}.$$

Evidently, $\mathcal{S}$ is a filter on ω . If A is a selector for $\mathcal{R}$ then $\omega - A \in \mathcal{S}$. A set $A \subseteq \omega$ is growing if and only if $\omega - A \notin \mathcal{S}$. Moreover if a filter $\mathcal{V}$ contains $\mathcal{S}$ as a subset then $\mathcal{V}$ is not a Q-point and each element of $\mathcal{V}$ is a growing set.

Lemma 1. Let $A \subseteq \omega$ be growing, $f \in {}^{\omega}\omega$. Then there exists a growing set $B \subseteq A$ such that $\bar{B} > f$.

PROOF. Let $b_0 \in A$, $b_0 > f(0)$. If $b_0, b_1, \dots, b_k$, $k = 1 + \dots + (n-1)$ are already constructed, we choice an integer l such that $|A \cap R_l| \geq n$ and $\min A \cap R_l > f(k+n)$. Now, choice $b_{k+1}, \dots, b_{k+n} \in A \cap R_l$.

Evidently the set $B = \{b_k; k \in \omega\}$ is growing and $\bar{B} > f$.

q.e.d.

We shall use the forcing construction as it is explained in [3]. Thus if M is a transitive model of ZFC, P,$\leq$ is a partially ordered set in M and G is an M-generic filter on P then M[G] is the corresponding model of ZFC - the generic extension of M. The complete Boolean algebra containing P as a dense subset is denoted by RO(P). The model M[G] is obtained as the range of the G-interpretation i_G defined on the Boolean-valued model $M^{RO(P)}$. If it is clear which generic filter G is intended, we simply denote the interpretation by i. For a formula φ and Boolean functions $f_1, \dots, f_n \in M^{RO(P)}$, the Boolean value $\|\varphi(f_1, \dots, f_n)\|$ is also defined in [3] (see pp. 152-169). If Q is a notion of forcing in M[G], we denote by $P * Q$ the iterated forcing (see [3], pp. 232-237).

For a given filter $\mathcal{V}$ on ω, J.Cichon [2] has constructed a forcing $P(\mathcal{V})$ as follows. The set $P(\mathcal{V})$ consists of ordered triples $\langle p, \mathcal{A}, f \rangle$, where for some integer n, the following holds true:

1) $p \in {}^{n}2$, $\mathcal{A} \in [\mathcal{V}]^{<\omega}$, $f \in {}^{\mathcal{A}}\omega$.

2) if $X \in \mathcal{A}$, $i \notin X$ and $f(X) \leq i < n$, then $p(i) = 0$.

The order $\leq$ on $P(\mathcal{V})$ is defined in the following way:

$$\langle p, \mathcal{A}, f \rangle \leq \langle p', \mathcal{A}', f' \rangle \equiv p \supseteq p', \mathcal{A} \supseteq \mathcal{A}', f \supseteq f'.$$

One can easily show that $P(\mathcal{V})$ satisfies the countable chain condition.

Now, let us suppose that M is a transitive model of ZFC, $\mathcal{V} \in M$ is a filter. If G is an M-generic filter on $P(\mathcal{V})$ we denote

$$X(\mathcal{V}) = \{ n \in \omega ; (\exists \langle p, \mathcal{A}, f \rangle \in G) p(n) = 1 \}.$$

In [2] it is shown that for each $X \in \mathcal{V}$, $X(\mathcal{V}) - X$ is finite. Moreover, we obtain

Lemma 2. If every element of $\mathcal{V}$ is a growing set then $X(\mathcal{V})$ is also growing.

PROOF. Let us denote

$$\mathcal{E}_n = \{ \langle p, \mathcal{A}, f \rangle \in P(\mathcal{V}); (\exists k) | \{ i \in R_k ; p(i) = 1 \} | \geq n \}.$$

It suffices to show that $\mathcal{E}_n$ is a dense subset of $P(\mathcal{V})$.

Thus, assume $\langle p, \mathcal{A}, f \rangle \in P(\mathcal{V})$. Then the set $Y = \bigcap \mathcal{A} \in \mathcal{V}$ is growing. Let k be such that $|Y \cap R_k| \geq n$ and $\mathrm{dom}(p) \cap R_k = \emptyset$. We denote $p' = p \cup (Y \cap R_k \times \{1\})$. Then $\langle p', \mathcal{A}, f \rangle \leq \langle p, \mathcal{A}, f \rangle$ and $\langle p', \mathcal{A}, f \rangle \in \mathcal{E}_n$.

q.e.d.

The Martin axiom MA and MA_{κ} is formulated e.g. in [4].

§2. Proof of the theorem A. For to obtain the model N we shall iterate the $P(\mathcal{V})$-forcing continuum many times. During the iteration we will construct a rapid filter that contains no selector for the partition $\mathcal{R}$.

For the iteration we need a well-known trick of enumerating all possible functions from ω into ω inside the resulting model N. Similar case of this trick is described with all details e.g. in [6].

Let P be a partially ordered set satisfying the countable chain condition, $|P| \leq 2^{\aleph_0}$. Then there exists a function H_P defined on $2^{\aleph_0}$ such that the range of H_P is the set of all Boolean functions $h \in M^{RO(P)}$ for which $\| h \in {}^{\omega}\omega \| = 1$. The value $H_P(\xi)$ will be denoted $H(P, \xi)$.

Let F be a fixed map of $2^{\aleph_0}$ onto $2^{\aleph_0} \times 2^{\aleph_0}$. Let K, L be maps of $2^{\aleph_0}$ onto $2^{\aleph_0}$ such that $F(\xi) = \langle K(\xi), L(\xi)\rangle$. We can assume that $K(\xi) \le \xi$ for every $\xi \in 2^{\aleph_0}$.

Now, by the transfinite induction we shall construct sequences $\{P_\xi ; \xi \in 2^{\aleph_0}\}$, $\{\mathcal{V}_\xi ; \xi \in 2^{\aleph_0}\}$, $\{G_\xi ; \xi \in 2^{\aleph_0}\}$, $\{B_\xi ; \xi \in 2^{\aleph_0}\}$ such that

3) P_ξ satisfies the countable chain condition, $|P_\xi| \le 2^{\aleph_0}$;
4) $P_\xi \subseteq P_\eta$ for $\xi < \eta$;
5) G_ξ is an M-generic filter on P_ξ ;
6) $G_\xi \subseteq G_\eta$ for $\xi < \eta$;
7) $M[G_\xi] \models$ " $\mathcal{V}_\xi$ is a filter, $\mathcal{S} \subseteq \mathcal{V}_\xi$ " ;
8) $M[G_\xi] \models$ " $\mathcal{V}_\eta \subseteq \mathcal{V}_\xi$ " for $\eta < \xi$;
9) $M[G_\xi] \models$ "$B_\xi - X$ is finite for each $X \in \bigcup\{\mathcal{V}_\eta ; \eta < \xi\}$" ;
10) $M[G_\xi] \models \overline{B}_\xi > i_{G_\xi}(H(P_{K(\xi)}, L(\xi)))$.

The construction is simple. We set $P_0 = P(\mathcal{S})$, G_0 is any M-generic filter on P_0 . The set $X(\mathcal{S})$ is growing in $M[G_0]$ by the lemma 2. By the lemma 1 there exists a growing set $B_0 \subseteq X(\mathcal{S})$ such that $\overline{B}_0 > i(H(P_{K(0)}, L(0)))$. Let $\mathcal{V}_0$ denote the filter generated by B_0 and $\mathcal{S}$ - everything inside the model $M[G_0]$.

Similarly, if P_ξ, $\mathcal{V}_\xi$, B_ξ are already defined, we denote $P_{\xi+1} = P_\xi * P(\mathcal{V}_\xi)$. Let $G_{\xi+1}$ be any M-generic filter on $P_{\xi+1}$ extending G_ξ . By the lemma 1 there exists a growing set $B_{\xi+1} \subseteq X(\mathcal{V}_\xi)$ such that $\overline{B}_{\xi+1} > i(H(P_{K(\xi+1)}, L(\xi+1)))$. Let $\mathcal{V}_{\xi+1}$ be the filter generated by $B_{\xi+1}$ and $\mathcal{S}$, constructed inside the model $M[G_{\xi+1}]$.

For λ limit we denote $P_\lambda = \bigcup\{P_\xi ; \xi < \lambda\} * P(\bigcup\{\mathcal{V}_\xi ; \xi < \lambda\})$. $G_\lambda, B_\lambda, \mathcal{V}_\lambda$ are defined analogously.

Directly from the construction one can see that 4) - 10) are fulfilled. The condition 3) is fulfilled by the well-known lemma about C.C.C.-iteration (see[3], p. 235 or [6]).

Now, we set $P = \bigcup\{P_\xi ; \xi \in 2^{\aleph_0}\}$, $G = \bigcup\{G_\xi ; \xi \in 2^{\aleph_0}\}$ and $\mathcal{V} = \bigcup\{\mathcal{V}_\xi ; \xi \in 2^{\aleph_0}\}$. Since P satisfies the countable chain condition, one can easily see that $\mathcal{V}$ is a rapid filter. Since each element of $\mathcal{V}$ is a growing set, $\mathcal{V}$ is not a Q-point.

If we change the construction in such a way, that on every step $\mathcal{V}_\xi$ is an ultrafilter containing B_ξ and extending $\mathcal{S}$, then the resulting filter $\mathcal{V}$ is an ultrafilter and actually a P-point.

<u>§3. Proof of the theorem B.</u> Let us remind that $\mathcal{B}$ is a basis of the filter $\mathcal{V}$ iff $\mathcal{B} \subseteq \mathcal{V}$ and for every $A \in \mathcal{V}$ there exists a $B \in \mathcal{B}$ such that $B \subseteq A$.

<u>Lemma 3.</u> Assume MA_κ holds true. Let $\mathcal{V}$ be a filter such that

a) each element of $\mathcal{V}$ is a growing set;

b) $\mathcal{V}$ has a basis of cardinality at most κ.

Then there exists a growing set A such that A - X is finite for each $X \in \mathcal{V}$.

PROOF. Let $\mathcal{B} \subseteq \mathcal{V}$ be a basis, $|\mathcal{B}| \leq \kappa$. If $X \in \mathcal{B}$ we denote

$$\mathcal{C}_X = \{\langle p, \mathcal{a}, f \rangle \in P(\mathcal{V}); X \in \mathcal{a}\}.$$

Evidently $\mathcal{C}_X$ is a dense subset of $P(\mathcal{V})$. By MA_κ there exists a filter G on $P(\mathcal{V})$ such that G is $\{\mathcal{C}_X; X \in \mathcal{B}\} \cup \{\mathcal{E}_n; n \in \omega\}$-generic (the sets $\mathcal{E}_n$ were defined in the proof of the lemma 2). The set $A = X(\mathcal{V})$ is the desired growing set.

q.e.d.

Now, the proof of the theorem B is straightforward. Let $\mathcal{F} = \{f_\xi; \xi \in \lambda\}$ be a dominating family. By the lemma 1, there exists a growing set B_0 such that $\overline{B}_0 > f_0$. Let us suppose that B_ζ is defined for $\zeta < \xi$. Let $\mathcal{V}_\xi$ be the filter generated by $\{B_\zeta; \zeta < \xi\}$. Then by the lemma 3 and the lemma 1, there exists a growing set B_ξ such that $\overline{B}_\xi > f_\xi$ and $B_\xi - B_\zeta$ is finite for $\zeta < \xi$. The filter $\mathcal{V}^\xi$ generated by $\{B_\xi; \xi < \lambda\}$ is rapid. Since each element of $\mathcal{V}$ is a growing set, $\mathcal{V}$ is not a Q-point.

<u>§4. Proof of the proposition</u>. Assume that $\mathcal{U}$ is a non-rapid filter. Then there exists a function $f \in {}^\omega\omega$ such that for each $g > f$ we have range $(g) \notin \mathcal{U}$. We can assume that f is strictly increasing. We set

$$a_0 = a_1 = \dots = a_{f(0)} = 1,$$
$$a_{f(0)+1} = \dots = a_{f(1)} = 1/\sqrt{2},$$
$$\vdots$$
$$a_{f(n)+1} = \dots = a_{f(n+1)} = 1/\sqrt{n+2},$$
$$\vdots$$

If $\overline{A} \not> f$ then for each k there exists an $n \geq k$ such that $\overline{A}(n) < f(n)$. Then

$$\sum_{i=0}^{n} a_{\overline{A}(i)} \geq (n+1).a_{\overline{A}(n)} \geq (n+1)a_{f(n)} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}.$$

Therefore $\sum_{n \in A} a_n = \sum_{n=0}^{\infty} a_{\overline{A}(n)} = +\infty$. Hence, if $\sum_{n \in A} a_n < +\infty$ then $\overline{A} > f$ and $A = \text{range}(\overline{A}) \notin \mathcal{U}$.

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