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FURSTENBERG's THEOREM IN TOPOLOGICAL DYNAMICS

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Abstract. In this paper we present an introduction in (abstract) topological dynamics. In an introductory section we describe the underlying ideas of this field of mathematical research. We illustrate this with the notion of almost periodicity and its translation into topological and algebraic terms. Then, in four sections, we develop some important notions from topological dynamics, culminating in an outline of a proof of Furstenberg's famous theorem on the structure of distal flows.

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INTRODUCTION

In classical dynamics, the possible states of a physical system are usually represented by points in an open subset M of $\mathbb{R}^n$ for some $n \geq 1$, and the evolution in time of the system is governed by a set of (autonomous) differential equations of the form

(1) $$\dot{x}_i(t) = X_i(x_1(t),\dots,x_n(t)), \quad i = 1,\dots,n,$$

where the X_i are real-valued functions on M. Under suitable conditions on these functions X_i such a system has unique solutions which are defined on all of $\mathbb{R}$; to be more precise, then for every point $x \in M$ there exists a unique differentiable function $\phi_x : \mathbb{R} \to M$ such that

(2) $$\dot{\phi}_x(t) = X(\phi_x(t)) \quad \text{for all } t \in \mathbb{R}\,)^*$$

$)^*$ here $X := (X_1,\dots,X_n)$.

(i.e. ϕ_x is solution of the system (1)) and, in addition,

(3) $\phi_x(0) = x.$.

It is not difficult to show, that in this situation one has

(4) $\phi(s,\phi(t,x)) = \phi(s+t,x)$

for all $s,t \in \mathbb{R}$ and $x \in M$; here $\phi(t,x) := \phi_x(t)$. Moreover, the mapping $\phi\colon \mathbb{R}\times M \to M$ is continuous under the conditions, referred to above. The mapping

$$\phi^t\colon x \mapsto \phi(t,x) = \phi_x(t)\colon M \to M$$

is well-defined and continuous for every $t \in \mathbb{R}$, and these mappings ϕ^t have the following properties:

- $\phi^0 = 1_M$, the identity mapping of M (cf.(3) above);
- $\forall s,t \in \mathbb{R} : \phi^s \circ \phi^t = \phi^{s+t}$ (cf.(4) above).

From this, it is clear, that each $\phi^t\colon M \to M$ is a homeomorphism and that $(\phi^t)^{\leftarrow} = \phi^{-t}$)*. A pair $\langle M,\phi\rangle$, where $\phi\colon \mathbb{R} \times M \to M$ is a continuous mapping, satisfying the two conditions above, is called a *continuous flow*, or a *topological transformation group* (*with acting group* $\mathbb{R}$) , or an $\mathbb{R}$*-space*.

In the study of continuous flows (which may or may not be derived from a dynamical system in the way described above) one often considers the real parameter t as *time* and the points of M as moving along their *orbits* $\phi_x[\mathbb{R}]$: a point which is at time 0 in the position x will be in position $\phi(t,x)$ at time t. Now one is interested in the "geometric" properties of the set of orbits which reflect the behaviour of the points moving along them. In particular, recurrence properties and asymptotic behaviour of points are important. Thus, in *topological dynamics* one studies topological properties of the "phase portrait" (= picture of orbits in M). Part of this study is, to translate the important dynamical notions from classical dynamics into topological properties. We shall illustrate this with an example.

An important dynamical notion is almost periodicity. Let $\langle M,\phi\rangle$ be an $\mathbb{R}$-space . A point $x \in M$ is called *almost periodic* whenever for every $U \in \mathcal{V}_x$)+ the set of return-times of x in U,

$$A(x,U) := \{t \in \mathbb{R} : \phi(t,x) \in U\},$$

has gaps of bounded length: there exists $\ell > 0$ such that $\mathbb{R} = [0;\ell] + A(x,U)$.

The following notion is topological: a subset P of M is called *minimal* whenever $P \neq \emptyset$, P is closed and P is invariant (i.e. $\phi(t,x) \in P$ for all $x \in P$

)* If $f\colon X \to Y$ is a mapping, then the inverse mapping is denoted $f^{\leftarrow}$.

)+ The neighbourhood filter of a point x in a given space is denoted $\mathcal{V}_x$.

and $t \in \mathbb{R}$), and P is minimal under these properties, that is, if P' is a non-empty, closed, invariant subset of P, then P' = P. It is not difficult to show that the following properties of a subset P of M are equivalent:

(i) P is minimal;

(ii) $\forall x \in P : P = \overline{\phi[\mathbb{R}\times\{x\}]}$ (i.e. for every $x \in P$, the *orbit* $\phi[\mathbb{R}\times\{x\}]$ of x is dense in P);

(iii) $\forall U \subseteq P$: U open in $P \Rightarrow \phi[\mathbb{R}\times U] = P$.

The following theorem essentially states, that minimality is a topological description of the dynamical notion of almost periodicity:

THEOREM. *Let* $\langle M,\phi\rangle$ *be a continuous flow, where* M *is a locally compact Hausdorff space, and let* $x \in M$. *The following statements are equivalent*:

(i) x *is an almost periodic point in* M;

(ii) *the orbit-closure* $\overline{\mathbb{R}x}$ *of* x *is a compact minimal subset of* M.

PROOF. For convenience, we shall write ty instead of $\phi(t,y)$, PQ or P.Q instead of $\phi[P\times Q]$ ($t \in \mathbb{R}$, $P \subseteq \mathbb{R}$, $y \in M$, $Q \subseteq M$), etc.

(i) $\Rightarrow$ (ii): In order to show minimality of $\overline{\mathbb{R}x}$, it is sufficient to prove that $x \in \overline{\mathbb{R}y}$ for every $y \in \overline{\mathbb{R}x}$. So let $y \in \overline{\mathbb{R}x}$ and let $U \in \mathcal{V}_x$, U compact. By almost periodicity of x there is a real number $\ell > 0$ such that $\mathbb{R} = [0;\ell] + A(x,U)$; then

$$\mathbb{R}x = [0;\ell].(A(x,U).x) \subseteq [0;\ell].U.$$

As $[0;\ell].U$ is compact, it is closed, hence $\overline{\mathbb{R}x} \subseteq [0;\ell].U$. First, this implies, that $\overline{\mathbb{R}x}$ is compact and second, it follows, that $y \in [0;\ell]\,U$, whence $\mathbb{R}y \cap U \neq \emptyset$. Since this holds for every $U \in \mathcal{V}_x$(U compact, but M is locally compact!) this shows that $x \in \overline{\mathbb{R}y}$.

(ii) $\Rightarrow$ (i): Let $V \in \mathcal{V}_x$, V open. By the equivalence of (i) and (iii) just above the theorem, we have $\overline{\mathbb{R}x} \subseteq \mathbb{R}V = \bigcup_{t\in\mathbb{R}} tV$. As each tV is open in M ($\phi^t : M \to M$ is a homeomorphism), compactness of $\overline{\mathbb{R}x}$ implies that there is a *finite* subset F of $\mathbb{R}$ such that $\overline{\mathbb{R}x} \subseteq FV$. If ℓ is the diameter of $F \cup \{0\}$, then it follows, that $\mathbb{R} = [0;\ell] + A(x,V)$, i.e. x is an almost periodic point. $\square$

The first to formulate and solve problems of dynamics as problems in topology was H. POINCARÉ. A systematic approach was undertaken for the first time by G.D. BIRKHOFF. With the introduction of important notions like minimal sets, recurrent and central motions he founded the topological theory of dynamical systems. This theory was developed, among others, by A.A. MARKOV, G.F. HILMY, M.V. BEBUTOV, V.V. NEMYTSKIĬ and V.V. STEPANOV. See [NS]; also [BS] and [S].

In the late fourties, the "classical" theory of dynamical systems was generalized by W.H. GOTTSCHALK and G.A. HEDLUND to the setting of arbitrary topological transformation groups. A *topological transformation group* (ttg) is a triple

$\langle T,X,\pi\rangle$ where T is a topological group, X is an arbitrary non-empty topological space and π is a continuous mapping from $T \times X$ into (in fact: onto) X, satisfying the following conditions)*

- $\pi^e = 1_X$ (e is the unit element of G)
- $\forall s,t \in T$: $\pi^s \circ \pi^t = \pi^{st}$.

From this, it is clear that each $\pi^t: X \to X$ is a homeomorphism of X onto itself, and that $(\pi^t)^{\leftarrow} = \pi^{t^{-1}}$. In this context, T is called the *phase group*, X is called the *phase space*, and π is called the *action* of the ttg$\langle T,X,\pi\rangle$. Usually in the discussions the phase group T is fixed, and then we shall denote the ttg $\langle T,X,\pi\rangle$ just by $\langle X,\pi\rangle$, and we shall call both the pair $\mathcal{X} := \langle X,\pi\rangle$ and the space X *a T-space*. Now *topological dynamics* can be defined as the study of ttg's with respect to those topological properties whose prototype occurred in classical dynamics. For example, a point x in a T-space $\mathcal{X}$ is called *almost periodic* whenever for every $U \in \mathcal{U}_x$ the set of "return-times"

$$A(x,U) := \{t \in T \,\vdots\, tx \in U\}$$

is *relatively dense* in T, i.e. there exists a compact set K in T such that $T = KA(x,U)$. Similar to the theorem above for continuous flows (=$\mathbb{R}$-spaces) one shows, that *if* X *is a locally compact Hausdorff space, then a point* x *in* X *is almost periodic iff its orbit-closure* $\overline{Tx}$ *is minimal and compact.* Here, of course, a subset M of X is called *minimal* whenever $M \neq \emptyset$, M is closed and invariant (i.e. $TM \subseteq M$), and M is minimal under these properties, that is, if A is a closed, invariant subset of X, $A \neq \emptyset$ and $A \subseteq M$, then $A = M$.

So by the result mentioned in the preceding paragraph, the study of compact minimal sets belongs to topological dynamics. (In fact, the classification of minimal sets is still one of the main unsolved problems in topological dynamics; cf. [Br]).

There is still one more step in the process of abstraction, which I shall now briefly indicate. Consider a T-space $\mathcal{X} = \langle X,\pi\rangle$ with X a compact Hausdorff space. Then the space X^X of all mappings (not necessarily continuous) of X into itself with the product topology is a compact Hausdorff space, and, with composition of mappings as multiplication, it is a *semigroup*. In this semigroup, all right translations $\xi \mapsto \xi\eta: X^X \to X^X$ ($\eta \in X^X$) are continuous. Moreover, the mapping $\bar{\pi} : t \mapsto \pi^t : T \to X^X$ is a continuous homomorphism of semigroups. Let $E := \overline{\bar{\pi}[T]}$ (closure in X^X). Then it is not difficult to see, that

)* Here, and in the sequel, we use the following notation: $\pi^t x := \pi(t,x) =: \pi_x t$ for $(t,x) \in T \times X$. Thus, we have mappings $\pi^t: X \to X$ and $\pi_x : T \to X$. However, usually we shall write tx instead of $\pi(t,x)$.

- E is a compact Hausdorff semigroup (a closed subsemigroup of X^X) such that for every $\eta \in E$ the mapping $\xi \mapsto \xi\eta\colon E \to E$ is continuous;
- the mapping $\bar{\pi}\colon T \to E$ is a homomorphism of semigroups with dense range, and the mapping $(t,\eta) \mapsto \bar{\pi}(t)\eta = \pi^t \circ \eta\colon T \times E \to E$ is continuous.

In general, a pair (ϕ,S), where S is a compact Hausdorff semigroup and $\phi\colon T \to S$ is a (continuous) homomorphism of semigroups, satisfying similar conditions as $(\bar{\pi},E)$ above is called an *enveloping semigroup* of T (the particular case of $(\bar{\pi},E)$ is called *the enveloping semigroup of* X). If (ϕ,S) is an enveloping semigroup, then it is clear, that the mapping $(t,\eta) \mapsto \phi(t)\eta\colon T \times S \to S$ is a continuous action of T on S, so that we obtain a T-space S. It is easy to see, that the *closed* invariant subsets of S are just the closed left ideals in the semigroup S (i.e. subsets M such that $SM \subseteq M$). Thus, *minimal subsets of S are just the minimal left ideals in* S (which turn out to be closed). A straightforward application of Zorn's lemma shows, that minimal left ideals always exist (as do minimal subsets in any compact Hausdorff T-space; cf. theorem 1.2 below).

Now dynamical properties of a compact Hausdorff T-space are, to a certain extend, reflected in the properties of its enveloping semigroup E. We shall illustrate this with the following example. We have seen above, that a point $x \in X$ is almost periodic if and only if its orbit-closure $\overline{Tx}$ is minimal (compactness need not be required as X is compact). This is reflected in E as follows:

<u>THEOREM</u>. *Let $X = \langle X,\pi\rangle$ be a* T*-space,* X *compact Hausdorff, and let* $x \in X$. *The following statements are equivalent*:

(i) $\overline{Tx}$ *is minimal* (*i.e.* x *is almost periodic*);

(ii) *there exists a minimal left ideal* M *in* E *such that* $x \in Mx$;

(iii) *every minimal left ideal* N *in* E *satisfies* $x \in Nx$;

(iv) *there exists a minimal left ideal* M *in* E *and an idempotent* $v \in M$ (*i.e.* $v^2=v$) *such that* $vx = x$;

(v) *every minimal left ideal* N *in* E *contains an idempotent* v *such that* $vx = x$.

<u>PROOF</u>.

The implications
$$\begin{tikzcd} & \text{(iv)} \arrow[dr] & \\ \text{(v)} \arrow[ur] \arrow[dr] & & \text{(ii)} \\ & \text{(iii)} \arrow[ur] & \end{tikzcd}$$
are obvious. For the proof that (ii) implies (i), observe that the evaluation-at-x, the mapping $\delta_x\colon \xi \mapsto \xi(x)\colon E \to X$ is a morphism of T-spaces (i.e. commutes with the actions in E and X: for every $t \in T$ we have $\delta_x(\pi^t\xi) = t\delta_x(\xi) = t\xi(x)$). It follows, that the minimal subset M of E is mapped onto a minimal subset M' of X. By hypothesis, $x \in M'$, hence $M' = \overline{Tx}$ and $\overline{Tx}$ is minimal. We shall now briefly indicate the proofs of the other implications.

(i) $\Rightarrow$ (iii): as δ_x maps E onto $\overline{Tx}$ and $\overline{Tx}$ is minimal, it follows that $\delta_x[N] = \overline{Tx}$ for every minimal ideal (= minimal invariant subset) N of E. Hence $x \in \delta_x[N] = Nx$.

(iii) $\Rightarrow$ (v): if $x \in Nx$, then $\{\xi \in N \,\vdots\, \xi(x) = x\}$ is a non-empty closed subsemigroup of E. It can be shown, that such a semigroup contains an idempotent (cf.

[E], lemma 2.9; also 3.7 below). □

The above theorem suggests, that a detailed study of the structure of minimal ideals in enveloping semigroups is important, and, of course, properties of idempotents. This study is usually performed in the so-called *universal enveloping* semigroup $(\iota,\beta T)$, which is characterized by the property that for each enveloping semigroup (ϕ,S) of T (so in particular, $(\bar{\pi},E)$) there exists a unique continuous homomorphism of semigroups $\bar{\phi}\colon \beta T \to S$ such that $\phi = \bar{\phi}\circ\iota$. It turns out, that $\iota\colon T \to \beta T$ is a dense embedding, so T can be viewed as a dense subgroup of the semigroup βT (in fact, βT is the Gelfand space of the C^*-algebra $RUC^*_u(T)$ of all bounded right uniformly continuous functions on T). It can be shown, that all minimal ideals in βT are mutually isomorphic as T-spaces, and that each of them serves as *universal minimal set*: if X is any minimal T-space, then there exists a (usually not unique) morphism of T-spaces from such an ideal onto X. Therefore, the study of minimal ideals in βT is the heart of many recent research in topological dynamics. This "abstract" topological dynamics was develloped by J. AUSLANDER, I.U. BRONŠTEĬN, R. ELLIS, H. FURSTENBERG and others. Cf. [GH], [E] and [Br]. In the last fifteen years there is a growing tendency in topological dynamics to study problems which are simular to problems in *ergodic theory* or which belong properly to ergodic theory (e.g. in connection with invariant measures). See for example [F] and W.A. VEECH [1977] and the references given there.

1. BASIC DEFINITIONS AND RESULTS

In the sequel, T shall always denote a topological group. In this chapter, we collect a number of generalities about compact Hausdorff T-spaces, their morphisms, and certain constructions which can be performed with them.

1.1. For the definition of a T-space $X = \langle X,\pi\rangle$ we refer to the Introduction. Although we shall usually write tx instead of $\pi(t,x)$, the homeomorphisms $y \mapsto ty\colon X \to X$ will be denoted by $\pi^t (t\in T)$, and the "evaluation mappings" $s \mapsto sx\colon T \to X$ by $\pi_x (x\in X)$.

Also the definition of invariant subset of X is given in the Introduction. If A is an invariant subset of X then also its closure $\bar{A}$ is invariant.

The minimal subsets of X are the minimal elements of the partial ordering (with respect to inclusion) of all non-empty closed invariant subsets of X (cf. the Introduction). Hence ZORN's lemma implies:

1.2. <u>THEOREM</u>. *Every non-empty compact invariant subset of* X *includes a minimal subset. In particular, if* X *is compact, then there is a minimal subset of* X. □

1.3. EXAMPLES.

(i) Let μ: $T \times T \to T$ denote the multiplication in T. Then $\langle T,\mu\rangle$ is a T-space. Since $Ts = T$ for every $s \in T$, T has no proper invariant subsets. In particular, T is minimal.

(ii) Let X be a topological space and let ξ: $X \to X$ be a homeomorphism. Then the formula

$$\pi(n,x) := \xi^n(x) = \xi\circ\xi\circ\ldots\circ\xi(x) \qquad (x\in X, n\in\mathbb{Z})$$

defines an action π of the additive group $\mathbb{Z}$ on X. In this case, the $\mathbb{Z}$-space $\langle X,\pi\rangle$ is called the *discrete flow, generated by* ξ, and in the sequel we shall denote it by $\langle X,\xi\rangle$ (although this notation is not consistent, it will cause no ambiguities, because $\pi^1 = \xi$ and, for each $n \in \mathbb{Z}$, $\pi^n = \xi^n = \xi\circ\ldots\circ\xi$). Note that every $\mathbb{Z}$-space $\langle X,\sigma\rangle$ is in this fashion generated by σ^1.

(iii) Let $X := S^1$ be the 1-torus, and let ξ: $X \mapsto X$ be the homeomorphism $z \mapsto az$ (here we identify S^1 with the set of all complex numbers of modulus one). It is well known that the discrete flow $\langle X,\xi\rangle$ generated by ξ is minimal $)^*$ iff *arg* a is an irrational multiple of π.

(iv) Let $X := S^1 \times S^1$ be the 2-torus, and define an action σ of $\mathbb{R}$ on X by the formula

$$\sigma^t(x,y) := (x\, exp(iat),\ y\, exp(ibt))$$

for $t \in \mathbb{R}$, $(x,y) \in X$. Then it is a well-known fact that the $\mathbb{R}$-space $\langle X,\sigma\rangle$ is minimal iff a/b is irrational.

(v) Let $X := \{0,1\}^{\mathbb{Z}}$; then X is a compact Hausdorff space (homeomorphic with the Cantor discontinuum). It is easy to see, that the "shift" σ: $X \to X$, defined by

$$(\sigma(x))_n := x_{n+1} \quad \text{for} \quad x = (x_n)_{n\in\mathbb{Z}} \in X$$

is a homeomorphism of X onto itself. The corresponding discrete flow $\langle X,\sigma\rangle$ is not minimal (e.g., the one-point set $\{(\ldots,0,0,0,\ldots)\}$ is invariant), but according to theorem 1.2 every orbit-closure in X contains a minimal set. Using the equivalence of almost periodicity and minimality as explained in the Introduction, it is easy to see that *for a point* $x \in X$ *the orbit-closure* $\overline{\mathbb{Z}x}$ *is minimal iff for every "block" in* x *of finite length there is a number* $\ell > 0$ *such that each block in* x *of length* ℓ *contains a copy of the given block* (a "block" in x of length j is a finite sequence of the form $x_i x_{i+1}\ldots x_{i+j-1}$ with $i \in \mathbb{Z}$). For example, consider the following sequence

$)^*$ A T-space $X = \langle X,\pi\rangle$ is called minimal whenever X is minimal.

$$\underbrace{\underbrace{\underbrace{\overset{\downarrow}{0}}_{B_1}\underbrace{1}_{B_1'}}_{B_2}\underbrace{10}_{B_2'}}_{B_3}\underbrace{1001}_{B_3'}10010110\ldots\ldots$$

Here a block B_i' denotes the "dual" of the block B_i (replace all zero's by one's and vice versa). If this sequence is reflected to the left, we obtain a point $x \in X$ such that its orbit-closure is minimal (this is a rather famous example, due to Marston MORSE, 1921).

1.4. A *morphism* of *T-spaces* $\phi: \mathcal{X} \to \mathcal{Y}$ is a continuous mapping $\phi: X \to Y$ such that $\phi(tx) = t\phi(x)$ for all $(t,x) \in T \times X$. If ϕ is surjective, then $\mathcal{Y}$ is called a *factor of* $\mathcal{X}$ (by ϕ) and $\mathcal{X}$ is called an *extension of* $\mathcal{Y}$ (by ϕ). An *isomorphism of* T-*spaces* is, of course, a morphism ϕ of T-spaces which is a homeomorphism of the underlying phase spaces; in that case, $\overset{\leftarrow}{\phi}$ is an isomorphism of T-spaces as well.

1.5. Let $\phi: \mathcal{X} \to \mathcal{Y}$ be a morphism of T-spaces. The following statements are easily verified:

(i) If A is an invariant subset of $\mathcal{X}$, then $\phi[A]$ is invariant in $\mathcal{Y}$; if B is an invariant subset of $\mathcal{Y}$, then $\overset{\leftarrow}{\phi}[B]$ is invariant in $\mathcal{X}$.

(ii) If A is a minimal subset of $\mathcal{X}$, then $\phi[A]$ is minimal in $\mathcal{Y}$ (consider $A \cap \overset{\leftarrow}{\phi}[B]$ for a closed invariant subset B of $\phi[A]$).

(iii) If $\mathcal{X}$ is minimal and ϕ is surjective, then $\mathcal{Y}$ is minimal.

(iv) If X is compact, Y is Hausdorff and $\mathcal{Y}$ is minimal, then ϕ is surjective (i.e. $\mathcal{Y}$ is a factor of $\mathcal{X}$) and Y is compact.

1.6. We shall now briefly describe the obvious constructions to form new T-spaces from given ones (subspaces, products, quotient and inverse limits). It will be clear without saying, that in all cases the canonical mappings (injections, projections, etc.) are morphisms of T-spaces.

(i) Let $\mathcal{X} = \langle X,\pi\rangle$ be a T-space and let Y be an invariant subset of X. Then the restriction of π to $T \times Y$ is an action of T on Y, which will, for convenience, also be denoted by π. Thus, we obtain the T-space $\langle Y,\pi\rangle =: \mathcal{Y}$. Notation: $\mathcal{Y} \subseteq \mathcal{X}$.

(ii) Let Λ be a set and let, for every $\lambda \in \Lambda$ a T-space $\mathcal{X}_\lambda = \langle X_\lambda,\pi_\lambda\rangle$ be given. Then the product $\mathcal{X} = \Pi_{\lambda\in\Lambda}\mathcal{X}_\lambda$ is defined as the T-space $\mathcal{X} = \langle X,\pi\rangle$, where

$$X := \prod_{\lambda\in\Lambda} X_\lambda \text{ (cartesian product with product topology);}$$

$$\pi(t,(x_\lambda)_{\lambda\in\Lambda}) := (\pi_\lambda(t,x_\lambda))_{\lambda\in\Lambda} \text{ for } t \in T \text{ and } (x_\lambda)_{\lambda\in\Lambda} \in X.$$

It is easily verified, that the thus defined mapping π is, indeed, an action of T on X.

(iii) Let $\mathcal{X} = \langle X,\pi\rangle$ be a T-space and let R be an invariant subset of $X \times X$

(action on $X \times X$ according to (ii) above) which is an equivalence relation on X. Then for every t in T a (continuous) mapping $\sigma^t: X/R \to X/R$ can be defined by

$$\sigma^t R[x] := R[\pi^t x] \quad (x \in X).$$

Thus we obtain a mapping $\sigma: T \times (X/R) \to X/R$ which satisfies the conditions of an action, except possibly continuity. If σ is continuous, then the T-space $\langle X/R, \sigma \rangle$ will be denoted by $\mathcal{X}/R$. In the following situation, continuity of σ is guaranteed:

Let X be *compact Hausdorff* and let R be an equivalence relation in X which is a *closed* invariant subset of $X \times X$. Then X/R is a compact Hausdorff space (cf. [Du], XI, 5.2) and $\sigma: T \times (X/R) \to X/R$ is continuous (this is, because the quotient map $q: X \to X/R$ is perfect: $1_T \times q$ is then perfect, hence a quotient mapping). Hence $\mathcal{X}/R$ is a T-space.

(iv) Let μ be a limit ordinal and let, for $\lambda < \mu$, $\mathcal{X}_\lambda$ be a *compact Hausdorff* T-space. Suppose that for every pair λ, ν of ordinals with $\nu < \lambda < \mu$ there exists a morphism of T-spaces $\phi_{\lambda\nu}: \mathcal{X}_\lambda \to \mathcal{X}_\nu$ such that $\phi_{\nu\kappa} \circ \phi_{\lambda\nu} = \phi_{\lambda\kappa}$ for $\kappa < \nu < \lambda < \mu$. Then the system $\{\mathcal{X}_\lambda \xrightarrow{\phi_{\lambda\nu}} \mathcal{X}_\nu\}_{0 \le \nu < \lambda < \mu}$ is called an *inverse system of* T-*spaces*. Since each of the spaces X_λ is compact, the subspace

$$Y := \{x = (x_\lambda)_{\lambda<\mu} \in \prod_{\lambda<\mu} X_\lambda : \forall \lambda, \nu (\nu < \lambda < \mu \Rightarrow \phi_{\lambda\nu} x_\lambda = x_\nu\}$$

of the full product is non-empty. In fact, Y is a closed, invariant subset of the product T-space $\mathcal{X} = \Pi_{\lambda<\mu} \mathcal{X}_\lambda$, hence we obtain a compact Hausdorff T-space $\mathcal{Y}$. The restrictions to Y of the canonical projections to the spaces X_λ will be denoted by $\phi_\lambda (\lambda<\mu)$. Clearly, each ϕ_λ is a morphism of T-spaces, $\phi_\lambda: \mathcal{Y} \to \mathcal{X}_\lambda$, such that $\phi_\nu = \phi_{\lambda\nu} \circ \phi_\lambda$ for $\nu < \lambda < \mu$. The T-space $\mathcal{Y}$, together with the morphisms $\phi_\lambda (\lambda<\mu)$ is called the *inverse limit* of the system $\{\mathcal{X}_\lambda \xrightarrow{\phi_{\lambda\nu}} \mathcal{X}_\nu\}_{0 \le \lambda < \nu < \mu}$. The following statement is easily verified: *if* $\mathcal{X}_\lambda$ *is a compact Hausdorff minimal* T-*space for every* $\lambda < \mu$, *then the* T-*space* $\mathcal{Y}$ *is also a minimal compact Hausdorff* T-*space* *).

2. CLASSIFICATION AND CONSTRUCTION OF MINIMAL SETS; EQUICONTINUITY

Let T be a given topological group. The two main unsolved problems in the theory of compact minimal sets for T are the *classification problem* (to classify all minimal sets according to isomorphism type) and the *construction problem* (to construct all minimal sets systematically). At the present, only partial answers can be given to these questions, even for discrete flows on compact metric spaces. We shall discuss some results, related to these two problems. *Unless stated*

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*) Thus, $\mathcal{Y}$ is not only the limit of the given inverse system in the category of all compact Hausdorff T-spaces, but also in the category of all *minimal* compact Hausdorff T-spaces.

otherwise, all T-*spaces will be compact Hausdorff.*

2.1. A T-space $X = \langle X,\pi\rangle$ is called *equicontinuous*)* whenever

$$\forall\alpha \in \mathcal{U}_X \ \exists\beta \in \mathcal{U}_X \ \vdots \ T\beta \subseteq \alpha.$$

Here $\mathcal{U}_X$ denotes the (unique) uniformity of X; as β is a subset of $X \times X$, the expression $T\beta$ makes sense (cf. 1.6(ii)). The equicontinuous minimal sets can be classified by means of the closed subgroups of the *Bohr compactification* $\gamma_T\colon T \to bT$ of T (here bT is a compact Hausdorff topological group, and γ_T is a continuous homomorphism with dense range, which is characterized by the following universal property: if $\psi\colon T \to G$ is a continuous homomorphism, where G is a compact Hausdorff topological group, then there exists a unique continuous homomorphism $\psi'\colon bT \to G$ such that $\psi = \psi'\circ\gamma_T$).

If H is a closed subgroup of bT, let $X := bT/H$ be the space of left cosets of H with the quotient topology. Then X is a compact Hausdorff space. Define $\pi\colon T \times X \to X$ by

$$\pi(t,x) := \gamma_T(t)sH \quad \text{if } x = sH \text{ for } s \in bT$$

($t\in T$). Then π is a continuous action of T on X and it is not difficult to show, that $X := \langle X,\pi\rangle$ is equicontinuous. In addition, X is minimal, because some point, hence (by equicontinuity) *every* point in X has a dense orbit in X.

Conversely, every equicontinuous minimal T-space can be obtained in this way. This is based on the following well-known lemma (which follows from results in [Bo], Chap.X, §3.5; see also [dV], 1.2.12):

2.2. LEMMA. *Let* $X = \langle X,\pi\rangle$ *be an equicontinuous* T-*space and let* E *denote the closure in* X^X *of the set of mappings* $\{\pi^t : t \in T\}$. *Then* E *is a group of homeomorphisms of* X, *and with the relative topology of* X^X *(which coincides with the topology of uniform convergence) it is a compact Hausdorff topological group.* □

2.3. If $X = \langle X,\pi\rangle$ is an equicontinuous T-space, then the mapping $\bar\pi\colon t \mapsto \pi^t\colon T \to E$ is a continuous homomorphism from T to a compact Hausdorff topological group (cf.2.2), so by the universal property of the Bohr compactification there exists a continuous homomorphism $\bar\pi'\colon bT \to E$ such that $\bar\pi = \bar\pi'\circ\gamma_T$. Fix $x_0 \in X$ and let

$$H := \{s \in bT \ \vdots \ \bar\pi'(s)(x_0) = x_0\}.$$

Then it is easy to check, that H is a closed subgroup of bT, and that the mapping

$$\phi\colon sH \mapsto \bar\pi'(s)(x_0) \ : bT/H \to X_0$$

)* Also the term *(uniformly) almost periodic* is used for this notion.

is a well-defined continuous injection. Since bT/H is compact and the orbit of x_0 is contained in the range of ϕ, ϕ is also surjective; hence ϕ is a homeomorphism of bT/H onto X_0. A straightforward computations shows, that ϕ is a morphism of T-spaces, where bT/H has the action described in 2.1. Thus, X is isomorphic with the T-space bT/H. The following diagram illustrates the arguments, given above:

$$\begin{tikzcd}
T \arrow[rr, "\gamma_T"] \arrow[d, "\pi_{x_0}"'] \arrow[dr, "\bar{\pi}"] & & bT \arrow[dl, "\bar{\pi}'"'] \arrow[d, "\text{quotient map}"] \\
X & E \arrow[l, "\delta_{x_0}"] & bT/H \arrow[l, "\phi"]
\end{tikzcd}$$

(here $\delta_{x_0}\colon \xi \mapsto \xi(x_0)\colon E \to X$ is the evaluation mapping at x_0).

Since for two closed subgroups H and H' of bT the spaces bT/H and bT/H' are isomorphic as T-spaces iff they are isomorphic as bT-spaces (bT acts on bT/H by $s(s'H) := ss'H$), and this is the case iff H and H' are conjugate, we obtain the following result:

2.4. THEOREM. *There is a* 1,1-*correspondence between isomorphism classes of equicontinuous minimal* T-*spaces and conjugacy classes of closed subgroups of the Bohr compactification* bT *of* T, *each equicontinuous minimal* T-*space being isomorphic with some* T-*space* bT/H, H *a closed subgroup of* bT. □

2.5. REMARK. The preceding theorem reduces the problem of classification and systematic construction of equicontinuous minimal T-spaces to the study of (conjugacy classes of) closed subgroups of the Bohr compactification of T. If T is locally compact and abelian, it follows from duality theory, that bT can be obtained as the character group of the charactergroup of T, the latter charactergroup provided with its *discrete* topology; thus, $bT = (T^\wedge)_d^\wedge$. Similarly, in that case all quotient spaces of the form bT/H, H a closed subgroup of bT, can be obtained as charactergroups of discrete subgroups of charactergroup of T, i.e. $G_d^\wedge$ with G_d a discrete subgroup of $T^\wedge$. For details, cf. [HR], Vol. 1. Of particular interest are, of course, the case that $T = \mathbb{Z}$ (discrete flows) and that $T = \mathbb{R}$ (continuous flows). In these cases, the spaces bT/H with H a closed subgroup of bT can also be characterized as the *monothetic* and *solenoidal* groups, respectively (again, for details cf. [HR], Vol.1).

From the point of view of topological dynamics, the equicontinuous minimal ttg's can be considered as essentially known, so that they can be used as the building blocks for other types of minimal flows.

2.6. For the systematic construction of minimal T-spaces, of the construction methods for new T-spaces from given ones (cf.1.6), only the formation of *quotients* and *inverse limits* can be used (and, if we start with a T-space which is not minimal, the formation of minimal subspaces). Indeed, it is only in special situations the case that a *product* of minimal T-spaces is minimal (if X and Y are minimal T-spaces and $X \times Y$ is minimal, then X and Y are called *disjoint*). Thus, one is forced to study equivalence relations in T-spaces and (in order to deal with inverse systems) surjective morphisms)*. We shall start with an equivalence relation, which is related to equicontinuity.

Let X be a T-space. Then the subset $Q(X)$ of $X \times X$ (often denoted just Q) is defined by

$$Q := \bigcap_{\alpha \in \mathcal{U}_X} \overline{T\alpha}$$

where the bar denotes closure in $X \times X$. Then Q is a closed, invariant subset of $X \times X$ such that $\Delta_X \subseteq Q$ and $Q = Q^{-1}$. There are examples where $Q \circ Q \nsubseteq Q$, so in general, Q is not an equivalence relation. Let $Q^{\#}(X)$ denote the smallest closed, invariant equivalence relation in X which contains $Q(X)$; $Q^{\#}(X)$ is called the *equicontinuous structure relation* in X. The reason for this name is (among others) that X is equicontinuous iff $Q = Q^{\#} = \Delta_X$ (indeed, if X is equicontinuous, then $\forall \alpha \in \mathcal{U}\ \exists \beta \in \mathcal{U} : T\beta \subseteq \alpha$, hence $Q = \bigcap_{\beta \in \mathcal{U}} \overline{T\beta} \subseteq \bigcap_{\alpha \in \mathcal{U}} \bar{\alpha} = \Delta$. Conversely, if X is not equicontinuous, then for some $\alpha \in \mathcal{U}$ we have $\overline{T\beta} \backslash \alpha \neq \emptyset$ for every $\beta \in \mathcal{U}$. Assuming α to be open, this implies that $\bigcap_{\beta \in \mathcal{U}} \overline{T\beta} \backslash \alpha \neq \emptyset$ by compactness, hence $Q \backslash \alpha \neq \emptyset$; in particular, $Q \neq \Delta$). This result can be generalized to the following

2.7. THEOREM. *Let $\phi: X \to Y$ be a factor. Then the following statements are equivalent*:

(i) *Y is equicontinuous*;

(ii) $Q^{\#}(X) \subseteq R_{\phi} := \{(x,x') \in X \times X : \phi(x) = \phi(x')\}$, *or equivalently*, $Q(X) \subseteq R_{\phi}$.

Consequently, the quotient mapping $q: X \to X/Q^{\#}(X)$ *is the largest equicontinuous factor of X, that is, $X/Q^{\#}(X)$ is equicontinuous, and a factor $\psi: X \to Z$ is equicontinuous iff there exists a morphism of* T-*spaces* $\tilde{\psi}: X/Q^{\#}(X) \to Y$ *such that* $\psi = \tilde{\psi} \circ q$.

PROOF.

(i) ⇒ (ii): If Y is equicontinuous, then by the statement just before the theorem, $Q(Y) = \Delta_Y$. As ϕ is uniformly continuous, for every $\beta \in \mathcal{U}_Y$ there exists

)* In fact, these are two sides of the same coin: if we form a quotient according to 1.6(iii), then the quotient mapping is a surjective morphism of T-spaces. Conversely, if $\phi: X \to Y$ is a surjective morphism of T-spaces, then Y is isomorphic to the quotient T-space X/R_{ϕ}, where $R_{\phi} := \{(x,x') \in X \times X : \phi(x) = \phi(x')\}$ is a closed, invariant equivalence relation in X.

$\alpha \in \mathcal{U}_X$ such that $\phi \times \phi[\alpha] \subseteq \beta$. Hence

$$\phi \times \phi\left[\bigcap_{\alpha\in\mathcal{U}_X} \overline{T\alpha}\right] \subseteq \bigcap_{\alpha\in\mathcal{U}_X} \overline{\phi\times\phi[T\alpha]} = \bigcap_{\alpha\in\mathcal{U}_X} \overline{T(\phi\times\phi[\alpha])} \subseteq \bigcap_{\beta\in\mathcal{U}_Y} \overline{T\beta}\ ,$$

that is, $\phi \times \phi[Q(X)] \subseteq Q(Y)$. Since $Q(Y) = \Delta_Y$, this implies that $Q(X) \subseteq R_\phi$. Since R_ϕ is a closed invariant equivalence relation in X, the latter inclusion is equivalent with $Q^{\#}(X) \subseteq R_\phi$.

(ii) $\Rightarrow$ (i): Obviously, it is sufficient to show, that $\phi \times \phi[Q(X)] = Q(Y)$. For the case, that Y is minimal or that ϕ is an open mapping, this is not so deep. The general case is a rather deep result, which needs the converse of lemma 2.2 (this converse is a corollary of a profound result of R. ELLIS). Cf. [Br], Lemma 1.6.15.

The final statement in the theorem should be clear now. Notice, that $X/Q^{\#}(X)$ is a T-space by 1.6(iii). □

2.8. REMARK. In view of the preceding theorem, we would like to know what $X/Q^{\#}(X)$ looks like. For example, under which conditions is $X/Q^{\#}(X)$ trivial (i.e. a one-point space), that is, $Q^{\#}(X) = X \times X$? In that case, X has no non-trivial equicontinuous factors. A related question, which turns out to be of great importance, is, under which conditions $Q^{\#}(X) = Q(X)$. For details, see for instance R. ELLIS & H. KEYNES [1971] or D. McMAHON & T.S. WU [1980]. See also 4.2 below.

3. PROPERTIES OF MORPHISMS OF (MINIMAL) T-SPACES

We shall now discuss some special types of morphisms of T-spaces. This brings us to the distinction between *absolute* and *relative* properties. If (E) is a property, applicable to morphisms of T-spaces, and for a given T-space X the morphism $X \to (*)$ ($(*)$ is the trivial one-point T-space) has property (E), then we say that X satisfies the *absolute version* of (E). Conversely, if (E') is a property of T-spaces, then it is often possible to formulate a property (E") for morphisms of T-spaces, such that (E') is the absolute version of (E"). In that case, (E") is called the *relative version* of (E'). In most cases, the process of relativation is as follows: suppose that property (E') for a T-space X can be formulated in terms of $X \times X$. Then the definition of the relative version (or: relativation) (E") of (E') is obtained by replacing the space $X \times X$ in the definition of (E') by the closed invariant subset $R_\phi := \{(x,x') \in X \times X \,\vdots\, \phi(x) = \phi(x')\}$ of $X \times X$; here $\phi\colon X \to Y$ is a morphism of T-spaces.

3.1. In accordance with the principles mentioned above an extension of T-spaces $\phi\colon X \to Y$ is called *equicontinuous* (or *almost periodic* in certain places in the literature) whenever

$$\forall\alpha \in \mathcal{U}_X\ \exists\beta \subseteq \mathcal{U}_X \,\vdots\, T(\beta\cap R_\phi) \subseteq \alpha$$

(compare with the definition of equicontinuity of X in 2.1).

3.2. EXAMPLE (H. FURSTENBERG [1963]).
Let $X := S^1 \times S^1$ be the 2-torus, and consider the discrete flow on X, generated by the homeomorphism ξ of X, where

$$\xi(x,y) := (x\ exp\ i\ \alpha,\ xy) \quad \text{for} \quad (x,y) \in X$$

(here we identify S^1 with $\{x \mid x \in \mathbb{C}\ \&\ |x| = 1\}$). Let $Y := S^1$, and consider the discrete flow on Y, generated by the homeomorphism η on Y, where

$$\eta(x) := x\ exp\ i\ \alpha \quad \text{for } x \in Y.$$

Finally, let $\phi: X \to Y$ be the projection onto the first coordinate: $\phi(x,y) := x$ for $(x,y) \in X$. Clearly, ϕ is a surjective morphism of discrete flows from $\langle X,\xi\rangle$ onto $\langle Y,\eta\rangle$. Since the restriction of ξ to each fiber $\overset{\leftarrow}{\phi}[x]$ $(x \in Y)$ is an isometry, it is clear that ϕ is an equicontinuous extension.

Observe, that η is an isometry of Y, so that the discrete flow (Y,η) is equicontinuous. On the other hand, $\langle X,\xi\rangle$ is not equicontinuous. Indeed, for $n \in \mathbb{Z}$ we have

$$\xi^n(x,y) = (x\ exp\ in\alpha,\ yx^n\ exp\ \tfrac{1}{2}in(n-1)\alpha).$$

Hence the points $\xi^n(x\ exp\ i\pi/n,y)$ are bounded away from the points $\xi^n(x,y)$ (the second coordinates of these points are each others opposite). As $(x\ exp\ i\pi/n,y) \to$ $\to (x,y)$ for $n \to \infty$, it follows, that the family $\{\xi^n \mid n \in \mathbb{Z}\}$ is not equicontinuous. So this example shows, that an equicontinuous extension of an equicontinuous flow need not be equicontinuous.

3.3. Before pointing out which property is preserved under equicontinuous extensions, we wish to consider the example above more closely. It is a particular case of the following situation.

Let $X = \langle X,\pi\rangle$ be a minimal T-space, and suppose that there exists also an action of a *compact Hausdorff group* K on X, denoted by $(x,k) \mapsto xk: X \times K \to X$)*, and satisfying the following conditions:

(i) the action of K commutes with the action of T, that is,
$\forall(t,x,k) \in T \times X \times K : (tx)k = t(xk)$;

(ii) the action of K on X is free, that is
$\forall(x,k) \in X \times K : k \neq e \Rightarrow xk \neq x$.

Then the equivalence relation

$$R_K := \{(x,x') \in X \times X \mid x' \in xK\}$$

)* so $x(k_1k_2) = (xk_1)\,k_2$ and $xe = x$

is T-invariant. According to 1.6(iii) one can define an action of T on the quotient space $X/R_K =: Y$, making the quotient mapping $\phi: X \to Y$ a morphism of T-spaces (continuity of the action of T on X/R_K can most easily be derived from the fact that ϕ is an open mapping). Now $\phi: X \to Y$ is called a *group extension of* Y (*by* K), and K is called the *structure group of* ϕ.

Observe, that in this situation for every $x \in X$ the mapping $k \mapsto xk : K \to X$ is injective, hence *it is a homeomorphism of the compact group* K *onto the* K-*orbit* xK *of* x *in* X. Since the K-orbits are just the fibers of the mapping $\phi: X \to Y$, it follows that all fibers are homeomorphic with K (in the example of 3.2, we have $K = S^1$).

3.4. PROPOSITION. *If* $\psi : Z \to Y$ *is a group extension* (Z *and* Y *minimal* T-*spaces*), *then* ψ *is equicontinuous.*

PROOF. Suppose ψ is not equicontinuous. Then

$$\exists \alpha \in \mathcal{U}_Z \quad \forall \beta \in \mathcal{U}_Z: T\beta \cap R_\psi \not\subseteq \alpha.$$

Hence for every $\beta \in \mathcal{U}_Z$ there exist points $z_\beta, z'_\beta \in X$ and $t_\beta \in T$ such that

$$(z_\beta, z'_\beta) \in \beta \cap R_\psi, \qquad t_\beta(z_\beta, z'_\beta) \notin \alpha.$$

Now for every $\beta \in \mathcal{U}_Z$ there exists $k_\beta \in K$ (where K is the structure group of ψ) such that $z'_\beta = z_\beta k_\beta$. Without loss of generality, we may suppose that the nets $(z_\beta)_{\beta \in \mathcal{U}_Z}$, $(t_\beta z_\beta)_{\beta \in \mathcal{U}_Z}$ and $(k_\beta)_{\beta \in \mathcal{U}_Z}$ converge in Z and K, respectively, say

$$z_\beta \rightsquigarrow z \in Z, \quad t_\beta z_\beta \rightsquigarrow y \in Z, \quad k_\beta \rightsquigarrow k \in K.$$

Then $z'_\beta = z_\beta k_\beta \rightsquigarrow zk$, and as $(z_\beta, z'_\beta) \in \beta$ for every $\beta \in \mathcal{U}_Z$, it follows that $z = zk$, hence $k = e$, the identity element of K. However,

$$t_\beta(z_\beta, z'_\beta) = (t_\beta z_\beta, (t_\beta z_\beta)k_\beta) \rightsquigarrow (y, yk) = (y,y) \in \Delta_Z$$

which contradicts the fact that the net $(t_\beta(z_\beta, z'_\beta))_{\beta \in \mathcal{U}_Z}$ is outside the neighbourhood α of Δ_Z. $\square$

3.5. Let $\phi: X \to Y$ be an arbitrary morphism of T-spaces. Then we define a subset Q_ϕ of R_ϕ by

$$Q_\phi := \bigcap_{\alpha \in \mathcal{U}_X} \overline{T\alpha \cap R_\phi}.$$

The smallest T-invariant closed equivalence relation in X containing Q_ϕ will be denoted $Q_\phi^\#$ (in the literature it is often denoted by E_ϕ). It is called the *relative equicontinuous structure relation* (of ϕ). Similar to the remark just above of theorem 2.7, we have here, that ϕ *is equicontinuous iff* $Q_\phi = \Delta_X$.

The following result (which will not be needed in the sequel) is the relativized version of theorem 2.4.

3.6. THEOREM. *Let* $\phi: X \to Y$ *be an extension of minimal* T-*spaces. Then the following statements are equivalent:*

(i) ϕ *is equicontinuous;*

(ii) *there exist a group extension* $\psi: Z \to Y$ *and a morphism of* T-*spaces* $\chi : Z \to X$ *such that* $\psi = \phi\circ\chi$)*.

PROOF.

(ii) ⇒ (i): this follows easily from 3.4 and the (not completely trivial) fact, that $\chi \times \chi[Q_\psi] = Q_\phi$ ("$\subseteq$" is obvious, but not relevant in this situation!). Cf. [Br], 3.13.2.

(i) ⇒ (ii): this proof makes use of a big machinery (which will not be developped in these notes), together with the famous but deep joint continuity theorem of R. ELLIS [1957]. For a proof of our theorem, see [Br], §3.17. A proof can also be based on §IX. 2 of [Gl]. □

3.7. REMARK. We wish to make a few remarks concerning the machinery, referred to in the proof above.

Recall from the introduction that there exists a universal minimal T-space M of which every minimal T-space is a factor. The underlying space M is obtained as a minimal left ideal in a certain enveloping semigroup of T; in particular, M itself has the structure of a semigroup. Some of the elementary facts about the structure of M are as follows (cf.[Gl]):

Let J denote the set of all idempotents in M; then $J \neq \emptyset$. In addition, it can be shown, that

- if $v \in J$, then $pv = p$ for all $p \in M$ (v serves as a right unit in M);
- if $v \in J$, then $vM := \{vp : p \in M\}$ is a subgroup of M;
- $M = \bigcup_{v\in J} vM$, and $vM \cap wM = \emptyset$ for $v,w \in J$, $v \neq w$.

So, in particular, M is the disjoint union of subgroups of the form vM ($v\in J$), which are usually *not* closed in M, so these groups are not compact. Important in the further study of M is the concept of the so-called τ-*topology*: a certain topology on each of these groups vM (weaker than the given topology) in which such a group vM is a compact T_1-space such that the multiplication in vM is separately continuous.

Now let X be a minimal T-space; then there exists a morphism of T-spaces $\gamma : M \to X$. Select an idempotent u in M and let $\gamma(u) =: x_0$. Then it turns out, that

)* In this situation, we say that ϕ is a *factor* of the group extension ψ.

$$G(X,x_0) := \{q \in uM \,\vdots\, \gamma(q) = x_0\}$$

is a τ-closed subgroup of uM. (Conversely, it can be shown, that every τ-closed subgroup of uM can be obtained in this way from some minimal T-space X.) If $\phi : X \to Y$ is a morphism of T-spaces and $y_0 := \phi(x_0)$, then it is easy to see, that $G(X,x_0) \subseteq G(Y,y_0)$ (where $G(Y,y_0)$ is defined, using the morphism $\phi\circ\gamma\colon M \to Y$; so $G(Y,y_0) := \{q \in uM \,\vdots\, \phi\gamma(q) = y_0\}$). These groups - the so called *Ellis groups* - play an important role in the study of morphisms of minimal T-spaces. For example, if $\phi : X \to Y$ is a group extension, then it can be shown, that $G(X,x_0)$ is a normal subgroup of $G(Y,y_0)$. Moreover, then there exists a homomorphism

$$\kappa : G(Y,y_0) \to \mathit{Aut}\, X \;)^*\,,$$

such that

$$\kappa(q)(\gamma(p)) = \gamma(pq) \quad \text{for } p \in M,\ q \in G(Y,y_0).$$

The kernel of κ turns out to be $G(X,x_0)$. Moreover, in this case (namely, that ϕ is a group extension), the range $\kappa[G(Y,y_0)]$ of κ is exactly the structure group K of ϕ (which can, of course, be viewed as a group of automorphisms of X) and, with respect to the τ-topology on $G(Y,y_0)$, the homomorphism $\kappa : G(Y,y_0) \to K$ is continuous. Hence the induced isomorphism

$$\kappa : G(Y,y_0)/G(X,x_0) \to K$$

is a continuous isomorphism of a compact group onto a compact Hausdorff group, hence a homeomorphism.

It should be clear now, that in the proof of (i) $\Rightarrow$ (ii) in the previous theorem, where a certain group extension has to be constructed, Ellis groups play an important role. The joint continuity theorem of ELLIS, referred to in the proof of 3.6 is needed in order to prove that the group constructed is, indeed, a topological group which acts continuously on the space under consideration.

3.8. If $\phi : X \to Y$ is a morphism of T-spaces, then $Q_\phi \subseteq R_\phi$, hence $Q_\phi^\# \subseteq R_\phi$, because R_ϕ is a closed invariant equivalence relation. Consequently, ϕ can be decomposed as follows

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\kappa"'] & & Y \arrow[dl, "\tilde{\phi}"] \\ & X/Q_\phi^\# & \end{tikzcd}$$

)* *Aut* X is, of course, the group of all T-automorphisms of X: all isomorphisms of T-spaces from X onto X.

Here $X/Q_\phi^\#$ is a well-defined T-space by 1.6(iii), κ is the quotient map, and $\tilde{\phi}$ the induced map. Notice, that both κ and $\tilde{\phi}$ are morphisms of T-spaces. Due to the following theorem, $\tilde{\phi}$ is called the *largest equicontinuous extension of Y which is a factor of* ϕ. (Observe, that this theorem is the relativized version of theorem 2.7).

3.9. THEOREM. *Let $\phi : X \to Y$ be a morphism of minimal* T*-spaces, and suppose that ϕ is decomposed as follows*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\chi"'] & & Y \\ & Z \arrow[ur, "\eta"'] & \end{tikzcd}$$

where also Z is a minimal T-space. Then the following are equivalent:

(i) η *is equicontinuous*;

(ii) $Q_\phi \subseteq R_\chi$, *or equivalently*, $Q_\phi^\# \subseteq R_\chi$.

In particular, the extension $\tilde{\phi} : X/Q_\phi^\# \to Y$ in 3.8 *is equicontinuous, and in the above diagram, η is equicontinuous iff there exists a morphism $\nu : X/Q_\phi^\# \to Z$, making the following diagram commutative*:

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[d, "\kappa"'] \arrow[drr, "\chi"] & & Y \\ X/Q_\phi^\# \arrow[rr, dashed, "\nu"] \arrow[urr, "\tilde{\phi}"] & & Z \arrow[u, "\eta"'] \end{tikzcd}$$

PROOF. The equivalence of (i) and (ii) is clear from the remark just above of theorem 3.6 and the (not completely trivial) fact that $\chi \times \chi[Q_\phi] = Q_\eta$ (here minimality is essential; see [Br], 3.13.2). The remainder of the theorem should now be obvious. □

3.10. The study of Q_ϕ for an arbitrary morphism ϕ of minimal T-spaces is very important in topological dynamics. First, there is the problem, under which conditions $Q_\phi = Q_\phi^\#$ (i.e. Q_ϕ is itself already an equivalence relation). Moreover, one is interested in characterizations of the extreme cases, $Q_\phi = \Delta_X$ (i.e. ϕ is equicontinuous) and $Q_\phi^\# = R_\phi$ (in which case no equicontinuous extension of Y can be a factor of ϕ). The importance of these questions will be illustrated below.

To this end, we have to introduce some new notions.

3.11. Two points x,y in a T-space are called *proximal* whenever $\overline{T(x,y)} \cap \Delta_X \neq \emptyset$. In that case, (x,y) is called a *proximal pair*, and the set of all proximal pairs in $X \times X$ is denoted P(X), or just P. So

$$P := \{(x,y) \in X \times X \mid \overline{T(x,y)} \cap \Delta_X \neq \emptyset\}.$$

For $x \in X$ the set $P[x] := \{y \in X : (x,y) \in P\}$ is called the *proximal cell* of x. It is clear, that $\Delta_X \subseteq P$ (i.e. $x \in P[x]$) and that $P = P^{-1}$ (i.e. $y \in P[x] \Leftrightarrow x \in P[y]$), but usually $P \circ P \nsubseteq P$.

By compactness of X, it is easy to prove that

$$P = \bigcap_{\alpha \in \mathcal{U}_X} T\alpha;$$

in particular, it follows, that P is T-invariant.

The T-space X is called *proximal* whenever $P = X \times X$, i.e. every pair of points in X is proximal to each other. Two points x,y in X are called *distal*, and (x,y) is called a *distal pair* whenever they are either equal or not proximal. The T-space X is called *distal* whenever $P = \Delta_X$, i.e. every pair of points in X is distal.

A *morphism of* T-*spaces* $\phi : X \to Y$ is called *proximal* (resp. *distal*) whenever every pair $(x,y) \in R_\phi$ is proximal (resp. distal). Thus, if we put

$$P_\phi := P(X) \cap R_\phi = \bigcap_{\alpha \in \mathcal{U}_X} T\alpha \cap R_\phi$$

then ϕ is proximal (resp. distal) iff $P_\phi = R_\phi$ (resp. $P_\phi = \Delta_X$). The notion of distallity is a generalization of the notion of equicontinuity. Indeed:

3.12. PROPOSITION. *Every equicontinuous* T-*space* X (*resp. equicontinuous morphism* $\phi : X \to Y$) *is distal.*

PROOF. Clear from the inclusions $P(X) \subseteq Q(X)$ (resp. $P_\phi \subseteq Q_\phi$) and the remarks just above 2.7 (resp. 3.6). □

3.13. The converse of this proposition is not true, not even if X is minimal. An example of a minimal, distal but not equicontinuous T-space X is provided by the flow on the 2-torus, described in 3.2. Indeed, in 3.2 it was shown, that X is not equicontinuous. Minimality follows, basically, from 1.3(iii). Distality follows from the observation that each fibre of the projection ϕ onto the first coordinate is transformed isometrically under ξ (pairs of points in the same fibre remain at the same distance under the flow), whereas different fibres also remain the same distance apart under the flow (so two points on distinct fibres can never get closer than the distance between these fibres).

3.14. If $\phi : X \to Y$ is a morphism of T-spaces, then it follows from uniform continuity of ϕ that

$$\phi \times \phi[P(X)] \subseteq P(Y),$$

that is, *each proximal pair of points is mapped onto a proximal pair*. Using this, it is easy to see that *the composition of two distal morphisms of* T-*spaces is again distal*. By induction, a similar statement is valid for compositions of finitely many morphisms (even for compositions of infinitely many, in the sense described in 3.15(iv) below). As a particular application of the statement above, we infer (also using 3.12) that an equicontinuous extension of an equicontinuous T-space is distal; also distal extensions of equicontinuous T-spaces are distal.

3.15. The following statements are easy consequences from the definitions and the results in 3.14.

(i) If Λ is an index set and, for every $\lambda \in \Lambda$, X_λ is a distal T-space, then $X := \Pi_{\lambda\in\Lambda} X_\lambda$ is distal.

(ii) If Y is a closed invariant subset of X and X is distal, then Y is distal.

(iii) If $\{X_\lambda \xrightarrow{\phi_{\lambda\nu}} X_\nu\}_{0\le\nu<\lambda<\mu}$ is an inverse system and $Y = \overleftarrow{\lim}_{0\le\lambda<\mu} X_\lambda$, and X_λ is distal for every λ, then so is Y (this follows easily from (i) and (ii)).

(iv) Let $\{X_\lambda \xrightarrow{\phi_{\lambda\nu}} X_\nu\}_{0\le\nu<\lambda<\mu}$ be an inverse system, and suppose that the following conditions are fulfilled

- for every ordinal λ with $0 \le \lambda < \mu$, $\phi_{\lambda+1,\lambda}: X_{\lambda+1} \to X_\lambda$ is distal;
- for every limit ordinal λ with $0 \le \lambda < \mu$ we have $X_\lambda = \overleftarrow{\lim}_{0\le\nu<\lambda} X_\nu$.

Then, if X_0 is distal, X_λ is distal for every ordinal λ under consideration (use transfinite induction). In particular, $Y := \overleftarrow{\lim}_{0\le\lambda<\mu} X_\lambda$ is distal.

4. THE STRUCTURE AND CONSTRUCTION OF MINIMAL DISTAL T-SPACES

4.1. If we consider successive minimal equicontinuous extensions of the trivial one-point T-space (*), and inverse limits thereoff, then we obtain *distal minimal* T-spaces (distallity: cf. 3.14(iv); minimality: cf.1.6(iv)).

Using the concept of "largest equicontinuous extension which is a factor of a given morphism" one may try to obtain every minimal distal T-space in this way. The idea is as follows. Let $\phi : X \to Y$ be a morphism of minimal T-spaces, and let μ be the first ordinal number with cardinality larger than the weight of X. Then by transfinite inductive definition an inverse system $\{X_\lambda \xrightarrow{\phi_{\lambda\nu}} X_\nu\}_{0\le\nu<\lambda<\mu}$ can be obtained, satisfying the following specifications:

- $X_0 = Y$, and for every ordinal $\lambda < \mu$ there is a morphism of minimal T-spaces $\phi_\lambda : X \to X_\lambda$ such that $\phi = \phi_{\lambda 0}\circ\phi_\lambda$;
- for every ordinal $\lambda < \mu$ the morphism $\phi_{\lambda+1,\lambda} : X_{\lambda+1} \to X_\lambda$ is the largest equicontinuous extension of X_λ which is a factor of ϕ_λ; thus, $X_{\lambda+1} = X/Q^{\#}_{\phi_\lambda}$, $\phi_{\lambda+1} : X \to X_{\lambda+1}$ is the corresponding quotient mapping, and the induced mapping $\phi_{\lambda+1,\lambda} : X_{\lambda+1} \to X_\lambda$ is equicontinuous (cf.3.8);
- if $\nu < \mu$ is a limit ordinal, then $X_\nu = \overleftarrow{\lim}_{0\le\lambda<\nu} X_\lambda$)*, and in addition, $\phi_\nu : X \to X_\nu$ is the morphism, induced by all mappings $\phi_\lambda : X \to X_\lambda$ with $\lambda < \nu$

)* implicitly, this means, that the mappings $\phi_{\nu\lambda}$ (from the "big" system) are the canonical projections in this inverse limit)

(i.e. ϕ_ν is the unique morphism such that $\phi_\lambda = \phi_{\nu\lambda}\circ\phi_\nu$ for every $\lambda < \nu$).
The following diagrams illustrate these conditions

$$\begin{tikzcd}
X \arrow[rrr, "\phi_0 = \phi"] \arrow[d, "\phi_{\lambda+1}"'] \arrow[dr, "\phi_\lambda"] \arrow[drrr, "\phi_1"] & & & X_0 = Y \\
X_{\lambda+1} \arrow[r, "\phi_{\lambda+1,\lambda}"] \arrow[urrr] & X_\lambda \arrow[r] \arrow[urr, "\phi_{\lambda,0}"] & \cdots \arrow[r] & X_1 \arrow[u, "\phi_{1,0}"'] \\
X/Q^{\#}_{\phi_\lambda} \arrow[u, equal] & & & X/Q^{\#}_{\phi} \arrow[u, equal]
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, "\phi_\nu"'] \arrow[dr, "\phi_\lambda"] & \\
\varprojlim_{0\le\lambda<\nu} X_\lambda = X_\nu \arrow[rr, "\phi_{\nu\lambda}"'] & & X_\lambda
\end{tikzcd}$$

Now we wish to look for conditions that the induced morphism of T-spaces from X to $\varprojlim_{0\le\lambda<\mu} X_\lambda$ (i.e. induced by all mappings $\phi_\lambda : X \to X_\lambda$ $(\lambda<\mu)$ is an isomorphism. The following reasoning shows, that in this construction we need not go further than the ordinal μ defined above:

The family $\{R_{\phi_\lambda} | \lambda<\mu\}$ is a descending family of closed subsets of R_ϕ. Since the cardinality of this family exceeds the weight of $X \times X$, it cannot consist of mutually different elements, hence there exists a first ordinal number $\gamma < \mu$ such that $R_{\phi\gamma} = R_{\phi\gamma+1}$. This implies, that

$$Q^{\#}_{\phi_\gamma} = R_{\phi_\gamma}$$

(indeed, $\phi_{\gamma+1} = \phi_\gamma$ and $\phi_{\gamma+1,\gamma}$ is the indentity mapping) and, by induction, that $X_\lambda = X_\gamma$ for all $\lambda \ge \gamma$ (and, of course, $\phi_\lambda = \phi_\gamma$ for $\lambda \ge \gamma$).

4.2. <u>THEOREM</u>. *If $\psi : Z_1 \to Z_2$ is a morphism of minimal* T*-spaces, and ψ is distal, then $Q^{\#}_\psi = Q_\psi$, that is, Q_ψ is already an equivalence relation. In particular, this is the case if Z_1 is distal.*

<u>PROOF</u>. This is a rather deep result; cf. W.A. VEECH [1977], Theorem 2.6.2. The proof presented there depends heavily on a profound study of the universal minimal set M and of Ellisgroups and the τ-topology (cf. 3.7 above). Roughly, it proceeds as follows: give each fiber $\overleftarrow{\psi}[z]$ $(z\in Z_2)$ a new topology (this topology is closely related to the τ-topology, mentioned in 3.7). Then it can be shown, that for $x \in \overleftarrow{\psi}[z]$ the set $Q_\phi[x]$ equals the intersection of the closures of the neighbourhoods of x in $\overleftarrow{\psi}[z]$ w.r.t. this topology (which may be not a T_1-topology). Using this, it can be shown, that the sets $Q_\phi[x]$ form a partition of X, i.e. Q_ϕ is an equivalence relation.

For the special case that Z_1 is a distal T-space, a shorter proof of the theorem is given in [Br], 2.9.7. □

4.3. <u>LEMMA</u>. *Let $\psi : Z_1 \to Z_2$ be a distal morphism of minimal* T*-spaces, and suppose* *that* Z_1 *is metrizable. If* $Q_\psi = R_\psi$*, then* ψ *is an isomorphism.*

PROOF. The uniformity of Z_1 has a countable base, say $\{\alpha_n : n \in \mathbb{N}\}$. Then we have

$$R_\psi = Q_\psi = \bigcap_{n=1}^{\infty} \overline{T\alpha_n \cap R_\psi}.$$

Assuming, that each α_n is open in $Z_1 \times Z_1$, this implies that each set $T\alpha_n \cap R_\psi$ is open and dense in R_ψ, hence BAIRE's theorem implies that $\bigcap_{n=1}^{\infty} T\alpha_n \cap R_\psi$ is dense in R_ψ:

$$R_\psi = \overline{\bigcap_{n=1}^{\infty} T\alpha_n \cap R_\psi} = \bar{P}_\psi.$$

However, ψ is distal, that is, $P_\psi = \Delta_{Z_1}$. Hence $R_\psi = \Delta_{Z_1}$, which means, that ψ is an isomorphism. □

4.4. Let $\phi : X \to Y$ be a morphism of minimal T-spaces. A *tower* for ϕ is an inverse system $\{X_\lambda \xrightarrow{\phi_{\lambda\nu}} X_\nu\}_{0 \le \nu < \lambda < \mu}$ such that the following conditions are satisfied
- $X_0 = Y$, $X = \varprojlim_{0 \le \lambda < \mu} X_\lambda$ and ϕ is the corresponding canonical projection of X onto X_0.
- for every limit ordinal $\lambda < \mu$ we have $X_\lambda = \varprojlim_{0 \le \nu < \lambda} X_\nu$.

If there exists a tower for ϕ with the additional property that for each $\lambda < \mu$ the morphism $\phi_{\lambda+1,\lambda} : X_{\lambda+1} \to X_\lambda$ has a property (E) (where (E) is a property, applicable to morphisms of T-spaces), then ϕ is called a *strict* E-*extension*. It is customary, to denote the property of being equicontinuous by I (from "isometrical"; cf. H. FURSTENBERG [1963]). Thus, a *strict* I-*extension* ϕ is a morphism ϕ for which there exists an I-tower: ϕ can be obtained as the inverse limit of equicontinuous extensions.

4.5. THEOREM. *Let* $\phi : X \to Y$ *be a distal morphism of minimal* T-*spaces, and let* X *be metrizable. Then* ϕ *is a strict* I-*extension.*

PROOF. Clear from 4.1 and 4.3. In fact, $\phi = \phi_{\gamma 0} \circ \phi_\gamma$ (γ as in 4.1), and from this it is clear, that ϕ_γ is distal; then 4.1 and 4.3 imply, that ϕ_γ is an isomorphism, so $X \cong X_\gamma$. Since it is obvious from the definition, that γ is a limit ordinal, we have an I-tower for ϕ. □

4.6. COROLLARY (H. FURSTENBERG [1963]). *Every metrizable distal minimal* T-*space is a strict* I-*flow, i.e. can be obtained by successive equicontinuous extensions of the trivial* T-*space and inverse limits thereoff.* □

4.7. REMARK. The result which we have formulated in Corollary 4.6 was the first in a long series of results of a similar nature, which form together the so called *structure theory* of minimal T-spaces and their morphisms.

The first generalization of FURSTENBERG's original result was proved by

R. ELLIS [1968]: he proved the relativized version of 4.6 as formulated in 4.5. (In fact he proved a slightly more general version: X need not be metrizable, but X must have enough metrizable factors to separate points; this condition is among others always fulfilled if T is a locally compact, σ-compact group.)

In order to describe the next development, we need some definitions. A morphism $\phi : X \to Y$ of T-spaces is called *point-distal* whenever there exists a point $x_0 \in X$ with $\overline{Tx_0} = X$)* such that $P_\phi[x_0] = \{x_0\}$ (i.e. all points in $\overleftarrow{\phi}[\phi x_0]$ are distal to x_0). Obviously, this is a generalization of a distal morphism (provided X has a point with dense orbit), and one might hope, that 4.5 could be generalized to point-distal extensions. In W.A. VEECH [1970] it was shown, that a point-distal extension of metrizable minimal T-spaces $\phi : X \to Y$ can be obtained *as a factor* of a strict PI-extension $\psi : Z \to Y$, as follows:

$$\begin{tikzcd} Z \arrow[r, "\theta"] \arrow[dr, "\psi"'] & X \arrow[d, "\phi"] \\ & Y \end{tikzcd}$$

This means, that ψ admits a tower as described in 4.4, where each $\phi_{\lambda+1,\lambda}$ is either proximal or equicontinuous. Here θ is a so-called *almost automorphic* extension: there exists a point $z_0 \in Z$ such that $\overleftarrow{\theta}[\theta z_0] = \{z_0\}$.)**

Actually, VEECH could prove this result only under an additional hypothesis (viz. the ϕ-distal points in X form a dense G_δ-set), but this condition could be removed from the hypothesis in R. ELLIS [1973].

An important question which came up remained unanswered for a long time: can the metrizability of the phase spaces (or related countability requirements like σ-compactness of T) be dropped? In R. ELLIS, S. GLASNER & L. SHAPIRO [1975], the following general structure theorem was proved. Let ϕ: $X \to Y$ be an arbitrary extension of minimal T-spaces. Then there exists a diagram as follows

$$\begin{tikzcd} Z_1 \arrow[r, "\text{proximal}"'] \arrow[d, "\eta"'] & X \arrow[d, "\phi"] \\ Z_2 \arrow[r, "\text{str.PI}"'] & Y \end{tikzcd}$$

Here all arrows are extensions of minimal sets, and η is such that $R_\eta = Q^{\#}_\eta$.

)* this condition is automatically fulfilled if X is minimal.

)** Clearly for each point of the orbit of z_0 the fiber for θ consists of one point, and these points form a dense subset of Z.

Under various conditions (including metrizability of X), η is an isomorphism. In particular, if Y is the trivial T-space (one point) and X is metric minimal such that the proximal cell $P[x_0]$ for some $x_0 \in X$ is countable, then both η and the proximal extension $Z_1 \to X$ in the diagram are isomorphisms: ϕ is strictly PI, i.e. X is a strict PI-flow.

In 1978, R. ELLIS obtained the first result for the non-metric case: FURSTENBERG's theorem for the non-metric minimal flow: *every minimal distal* T-*space* X *is a strict* I-*flow*. See R. ELLIS [1978].

Using ELLIS' techniques, D.C. McMAHON & L.J. NACHMAN [1980] proved the following non-metric verson of VEECH's structure theorem (absolute case): *every minimal point-distal* T-*space* X *is a factor of a strict* PI-*flow*.

Finally, D.C. McMAHON & T.S. WU [1981] proved the "relative FURSTENBERG" theorem (i.e. 4.5 above) for the non-metric case: *every distal extension of minimal* T-*spaces in strictly* I.

Thus, there is still one open case: the non-metric version of the VEECH structure theorem, relative case (i.e.: is it true that every point-distal extension of minimal sets is a factor of a strict PI-extension?)

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