On Permutographs

Gerhard G. Thomas

Summary: The mathematical analysis of many-valued logic and of its supplement, the logic of value-contextures*, leads to structures of permutations and problems of combinatorics. This exploration uses methods of graph theory. The fundamental operator of logic – the (many-valued) negation – was picked out to demonstrate something about the connection of many-valuedness and contextures. The network of negations of this logic are *permutographs*: regular graphs on permutations. Homogeneousness of knots (points) in connected simple graphs will introduced.

Introduction

The following graph theoretical structures and problems emerged in the context of questions of the many-valued logic of *Gotthard Günther*. In my context the focal point is put on the presentation of mathematical problems, only slight attention is given to many-valued logic, i.e. there will be only some short (technical) annotations concerning the field of many-valued logic. In many-valued logic – including also the case of n = 2 (that means the classic mathematical logic respectively the Aristotelian logic or predicative logic) – the focus is on n logical values. These values may not nessecarily be truth values (Wahrheitswerte).

In the approach of *Günther* these logical values are homogeneous and there exists no relation of subsumption or ordination between them and they are for the present independent.

$$\boxed{1}\ ,\ \boxed{2}\ ,\ \boxed{3}\ ,\ \dots\ ,\ \boxed{n}\ .$$

These values can be connected by logical operators. As we shall see soon the attributes of the logical values together with the negation operator N (negator) can be described by structures of permutations.

A *negator* N is a one-placed function, defined on two values x_1, x_2

$$N(x_1) = x_2 \text{ and } N(x_2) = x_1.$$

The logical variables p,q,r,... are defined on the range of logical values

$$1\ ,\ 2\ ,\ 3\ ,\ \dots\ ,\ n\ .$$

For n = 2 we get the well-known figure

p	N(p)
1	2
2	1

If n = 3, then we get already five different negations of p.

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*For detailed introduction into the philosophy of many-valued logic and contextures see [3], especially volume 3. Articles are written in english or german.

p	$N_1(p)$	$N_2(p)$	$N_1N_2(p)$	$N_2N_1(p)$	$N_1N_2N_1(p)$
1	2	1	2	3	3
2	1	3	3	1	2
3	3	2	1	2	1

p and its five negations correspond exactly to 3! permutations of degree 3. Also for n = 3 holds $N_iN_i(p) = p$.

If we transform logical negation problems into graph theoretical structures, we will see that exactly n - 1 negators are sufficient to generate all n! states of negation (including the identity).

Value-contextures generate permutographs

Be Π the set of n! permutations. The elements of Π correspond to the n! possible states of a n-valued logic of negations. The Negator N shall be an one-place relation on to values x_1, x_2

$$x_1 \overset{N}{\longleftrightarrow} x_2$$

$N(x_1) = x_2$ and $N(x_2) = x_1$ hold at the same time.

In the two-valued logic x_1 and x_2 stand for 'true' and 'false'. In combinatorics we call the exchange of two integers (or elements) *transposition*. When both the integers of a transposition t are in ascending order (i, i+1), then we name t *standard transposition*.

There are $\binom{n}{2}$ different transpositions. They form the set TR of transpositions. If we apply a qualified sequence of transpositions

$$t_{i_1}, t_{i_2}, t_{i_3}, \ldots, t_{i_r}$$

to a permutation π, so we get all of the other n!-1 permutations after a finite number of steps. In other words: From every permutation π you can construct the whole set Π of permutations by using the right transpositions. The following theorem says, that it is not nessesary to use all $\binom{n}{2}$ transpositions for the construction of Π.

A connected graph T is called *tree* , if the graph is connected and acyclic. A tree of n knots (points) has n-1 edges (lines).

THEOREM: Be X:= (1, 2, ..., n). A set T of n-1 transpositions generates the symmetric group S_n (with n! elements) if, and only if, the graph (X,T) forms a tree.

The proof of this theorem you will find, for example in *Berge* [1] .

Be (1) X:= (1,2,...,n) the set of n values;

(2) $\Pi := (\pi_1 | \pi_2 | \cdots | \pi_{p(r,n)})$ the set of permutations; P(r,n):= total number of permutations;

(3) $T := (t_1, t_2, \ldots, t_m)$ $2 \leq m \leq \binom{n}{2}$ a set of transpositions of elements of X;

(4) the *contexture* CT = (X,T) is a (connected) graph with |X| knots and |T| edges;

(5) Π is interpreted as the set of knots of graph PG;

(6) two knots π_i , $\pi_j \in \Pi$ are only then connected, if there exists a $t \in T$, which transforms

$$\pi_i \overset{t}{\longleftrightarrow} \pi_j ;$$

(7) all $t \in T$ are edges of the |Π| knots of a graph PG;

if (1) to (7) holds then PG = PG(Π,CT) is called a *permutograph*.

Remark: Every permutograph PG is m-regular.

Gerhard G. Thomas

On Permutographs

Negations-Mandala PG_1 (5)

Negations-Mandala PG_1 (5)

Erzeugungsbaum

1 —N_1— 2 —N_2— 3 —N_3— 4 —N_4— 5

generating tree

Negatoren

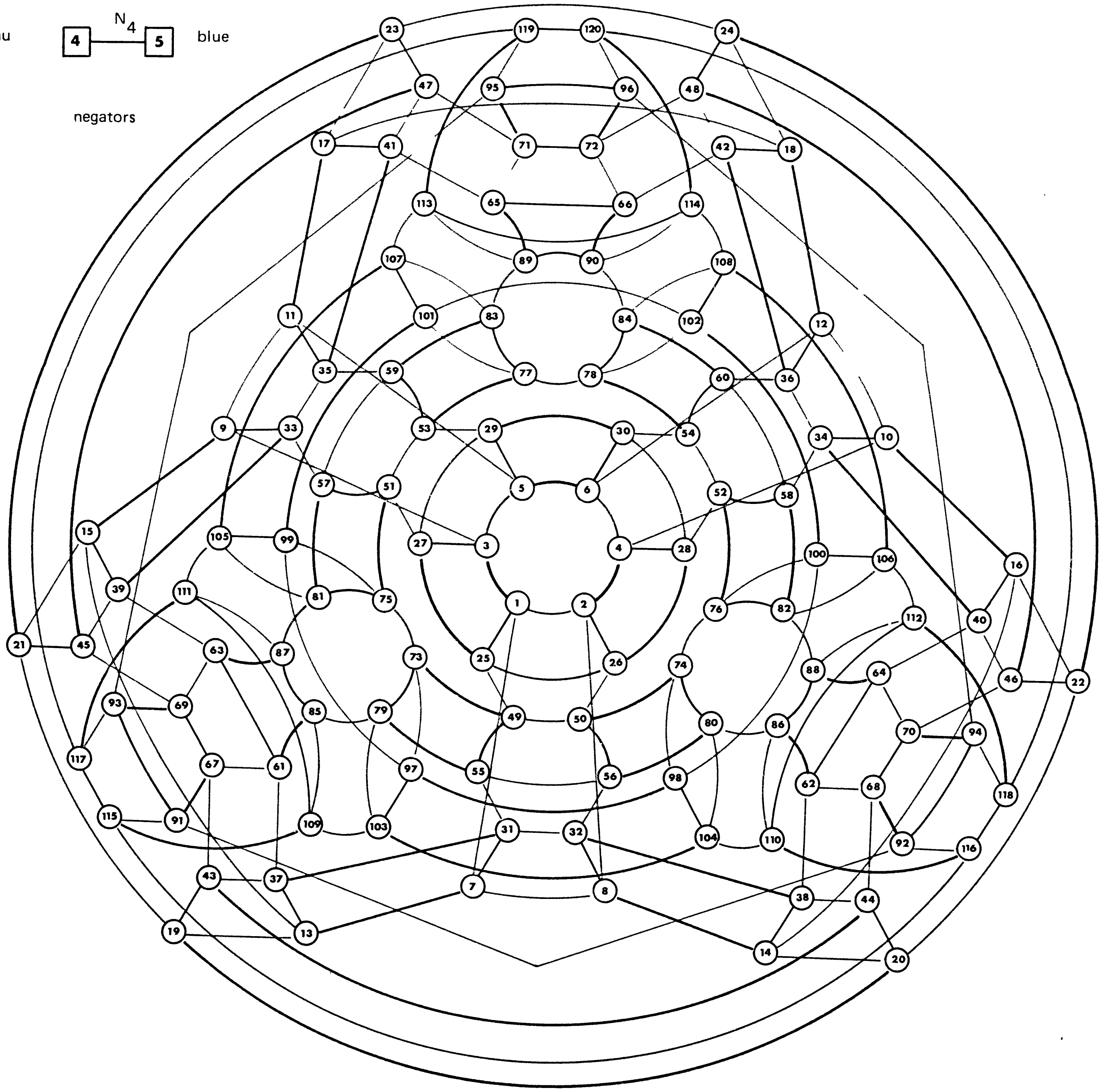
schwarz 1 —N_1— 2 black

rot 2 —N_2— 3 red

grün 3 —N_3— 4 green

blau 4 —N_4— 5 blue

negators



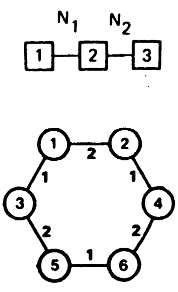
PG_1 (5) : Mandala of Negations

Herstellung: Klinikum Steglitz, AVD-Druckerei

Because theorem 1 holds, it is clear that n-1 transpositions are enough to connect all $\pi \in \Pi$. That means the minimal contexture CT is a tree. In this case the graph PG is (n-1)-regular.

EXAMPLE 1

① 123
② 132
③ 213
④ 231
⑤ 312
⑥ 321



tree-contexture of values 1,2,3 forms a *line*.
Negator N_1 changes 1 ⟷ 2
Negator N_2 changes 2 ⟷ 3

The tree-contexture describes the generating scheme of permutographs.

These sequences of negations form the identity:

$N_1N_2N_1N_2N_1N_2\,\pi = \pi$

$N_2N_1N_2N_1N_2N_1\,\pi = \pi$

Permutograph PG(|3!|,□-□-□)

In *Harary* [5] you find diagrams of unlabelled trees for n = 1,2,...,10. The following table is an extraction of *Sloane* [7].

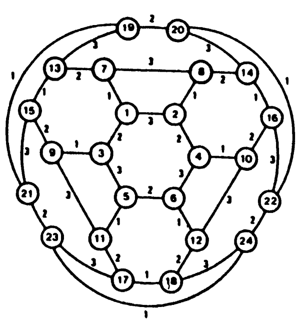
n	3	4	5	6	7	8	9	10	11	12	13	14
T(n)	1	2	3	6	11	23	47	106	235	551	1301	3159

number of different unlabelled trees

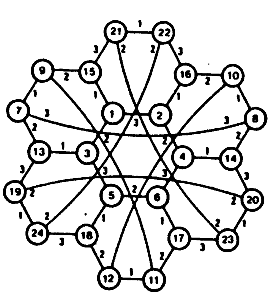
Cayley [2] found in 1889 that there exist n^{n-2} labelled trees. That means, for a given n there exist n^{n-2} different negation systems with a minimum number of negators (if the arrangement of the values is also regarded). Since all labelled trees generate isomorphic permutographs, it is sufficient to regard only *one* value contexture.

For n = 4 there exist 2 different tree-contextures and therefor 2 different permutographs on 4-permutations:

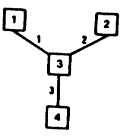
CT:= $T_1(4)$



PG(4!, $T_1(4)$)



PG(4!, $T_2(4)$)



CT:= $T_2(4)$

The coloured figure on the folded page is one possible geometrical representation of the permutograph PG(5!,□-□-□-□-□).

Basic cycles in permutographs PG with tree-contexture

In this chapter we regard permutographs with a tree-contexture CT = T(n). T(n) is a tree on n knots (values). The permutograph PG(II,T(n)) represents a network for $n > 2$, i.e. the net of negations of a n-valued logic. This net can be constructed from mashes (Maschen) of minimal length. These constistuting mashes are called *basic cycles* (Basiskreise),i.e. cycles without chords. Basic cycles always have the length of 4 or 6, if the contexture of values forms a tree.

THEOREM 1: Every tree-contexture T(n) (tree of negators) produces a permutograph PG containing $\binom{n-1}{2}$ different basic cycles.

Proof We label the n-1 edges of the tree of negators by $N_1, N_2, \ldots, N_{n-1}$. We differentiate neighboured und unneighboured edges of the tree-contexture T(n).
In logical terms we express identity as follows:

$$N_i N_j \ldots N_k(\pi) = \pi.$$

That means identity corresponds with a cycle in PG. In case of alternating indices of negators we get the identities after a sequence of 4 or 6 negators. Consequently we gain a basic cycle of length 6, if the edges of the tree-contexture T are neighboured and a basic cycle of length 4, if these edges are unneighboured.

6-cycle: $N_iN_jN_iN_jN_iN_j \equiv I$ $\quad$ (N_i, N_j neighboured)

4-cycle: $N_iN_kN_iN_k \equiv I$ $\quad$ (Ni, N_k unneighboured)

By matching all possible indices – neighboured or not – we get the total number of basic cycles $\binom{n-1}{2}$, because every graph,constituting a tree has n-1 edges. ■

There exist permutographs PG with basic cycles only of length 6. The corresponding tree T of these PG has the shape of a *star*, i.e. one knot of T is connected with all other (n-1) knots. Following theorem 1 there are $\binom{n-1}{2}$ different basic cycles of length 6.

If the tree-contexture T of PG forms a *line*

□-□-□- ··· □-□

we have $\binom{n-2}{2}$ basic cycles of length 4 and n-2 basic cycles of length 6; in total $\binom{n-2}{2} + n-2 = \binom{n-1}{2}$ basic cycles.

THEOREM 2. Be BC4(n) respectively BC6(n) the total number of basic cycles of length 4 respectively of length 6, d_i the number of knots with degree i of the tree-contexture T(n) of PG, then

$$BC6(n) = \sum_{i=2}^{r} \binom{i}{2} d_i \quad r := \text{maximal degree of a knot in } T(n)\ 2 \geqslant r \geqslant n-1$$

and

$$BC4(n) = \binom{n-1}{2} - BC6(n).$$

Proof: Basic cycles of length 6 corresponds with neighboured edges in T(n). A knot of degree i ($i > 1$) has exactly i neighbours in T(n). Each pair of neighboured edges constitutes a basic cycle of length 6

$$N_iN_jN_iN_jN_iN_j \ .$$

The number d_i of the matched neighboured edges of a knot with degree i is $\binom{i}{2}$. A knot of degree 1 has only one neighbour so that a matching of neighbours is impossible. It follows the number of

$$BC6(n) = \sum_{i=2}^{r} \binom{i}{2} d_i$$

basic cycles of length 6. Following theorem 1 there exists $\binom{n-1}{2}$ · BC6(n) basic cycles, so that

$$BC4(n) = \binom{n-1}{2} \cdot BC6(n). \blacksquare$$

Example: CT:= $T_2(6)$ (see suppl. tab. basic cycles), $d_1 = 3$, $d_2 = 2$, $d_3 = 1$; $BC6(T_2(6)) = 3\binom{1}{2} + 2\binom{2}{2} + \binom{3}{2} = 0+2+3=5$

COROLLARY 1: The total number of basic cycles of length 4 respectively length 6 be M4(n) resp. M6(n). If the value-contexture forms a tree, then

$$M4(n) = \frac{P(r,n)\ BC4(n)}{4} \qquad M6(n) = \frac{P(r,n)\ BC6(n)}{6}$$

Proof: PG(II,T) contains P(r,n) knots. Every knot k belongs to the BC4(n) basic cycles of length 4 and to the BC6(n) basic cycles of length 6. A basic cycle consists of 4 respectively 6 knots. Therefrom the result follows immediately. ■

Remark (quoted from *Fiorini/Wilson* [3]): In 1973 *Szekeres* introduced two-coloured cycles (this corresponds with the alternating indices) of cubic graphs and called it *basic circuits*. Each edge of G lies exactly on two of these basic circuits. That also holds for all edges of PG and PG must not be cubic.

Compositions and Decompositions of Permutographs by Unions or Subgraphs of Tree-contextures

Here are given only some small examples for the understanding of the effect of contextures. A more detailed analysis will appear later. The indices on edges of a permutograph are negator-indices. Integers in squares are names of values. Integers surrounded by cycles are ordinal numbers of permutations in lexicographic order.

1. Decompositions

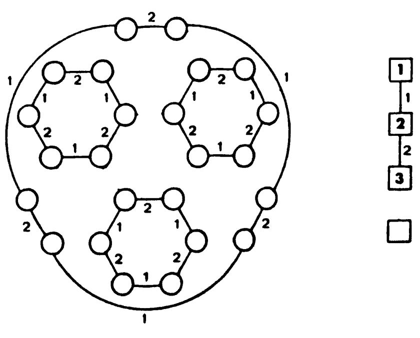
Because the tree-contextures are minimal contextures to generate a connected permutograph on *all* P(r,n) permutations, it is clear, that a contexture, which is a subgraph of such a tree-contexture leads to a decompositon of the P(r,n)-permutograph.

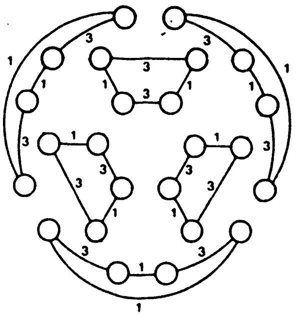
Example 1: PG_1(4!; □–□–□–□)

CT_1:= [1]–N1–[2]–N2–[3]–N3–[4]

f.e. is a sub-contexture of CT_1 CT_{1a}:= [1]–N1–[2]–N2–[3].

PG_{1a}:= PG(4! ; □–□–□) is a decomposition of PG_1 into 4 permutographs PG(3! ; □–□–□).





If you choose the subcontexture

sub CT_{1b}:= [1]–1–[2]   [3]–3–[4]

then you get a decompositon of PG_1 with 6 components, which are not permutographs (with a tree-contexture) of lower order; but the components have the shape of basic cycles

C_4:= (1313).

Example 2:

PG_2(5! ; □–□–□–□–□)

sub CT_{2a}:= [1]–1–[2] [4]–4–[5]

PG_2 is decomposed by CT_{2a} into 30 components of the shape

C_4:= (1414).

2. Compositions

The union of at least two contextures is called a (contexture-) *composition*.

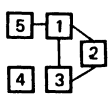
Example 3: Given PG_2:= PG(5! ; T_1(5)) ; CT_2:= T_1(5):= [1]–[2]–[3]–[4]–[5] .

A composition of two subgraphs of T_1(5) – for instance

[1]–[2]–[3]   and   [3]–[4]–[5]   leads to the complete contexture of PG_2.

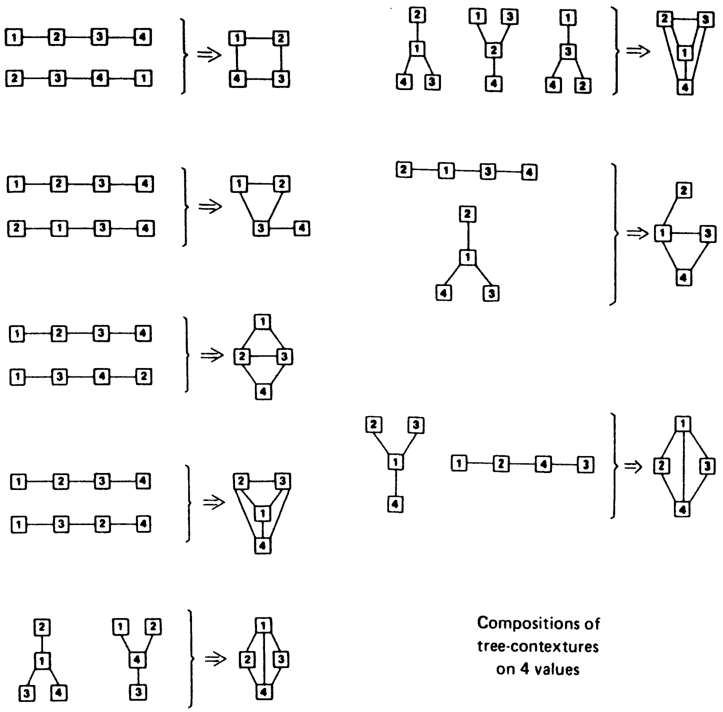
A permutograph PG is called	contexture CT
unbalanced	CT is disconected
balanced	CT is a tree (a minimal connected graph).
overbalanced	CT is a graph with cycles.

Example 4: CT_{3a}:= [1]–[2]–[3] CT_{3b}:= [1]–[3]–[5] .   The union $CT_{3a} \cup CT_{3b}$ is

CT_3:


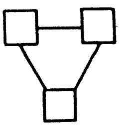
PG_3(5! ; CT_3) is an unbalanced permutograph with 5 components, each on different 24 permutations (5 x 24 = 120). Each component is an overbalanced permutograph on 4 values (values 1 - 5 except value 4): PG(4! ; CT_3/ 4).

The following contextures are possible compositions of the two tree-contextures for 4 values. It depends on the connections of values – not only on the (tree–) structure of the contextures, if you get a composed contexture with 4,5 or 6 different negations. Two line-contextures are sufficient for a *complete* contexture (i.e. all values are connected with all other values). By the unions of only star-contextures you need for the complete contexture 3 star-contextures. Of course are all such compositions overbalanced.



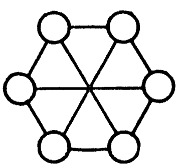
Compositions of tree-contextures on 4 values

The following two permutographs are overbalanced and compositions of minimal permutographs, which are shown above. Permutograph PG_4 has only basic cycles of length 4. This is a consequence of neighbourhood of the values in the cycle-contexture.

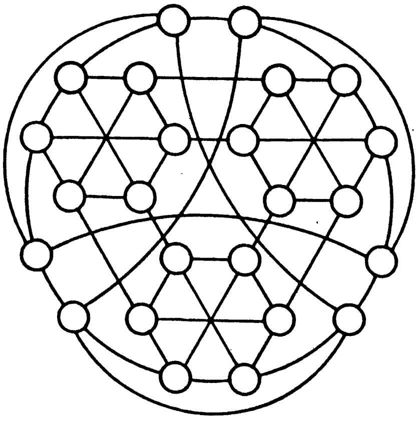


CT_4

PG_4 is the permutograph with the smallest cycle-contexture

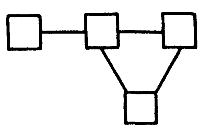


$PG_4 := PG(3!, CT_4)$



$PG_5 := PG(4!, CT_5)$

PG_5 has the structure of each of the 5 components in example 4



CT_5

Homogeneity of knots

By construction the permutographs have a certain equality in structure. They are very symmetrical. All knots of a permutograph, e.g. the permutations $\pi \in \Pi$, are of equal rank. Each knot is embedded in the same set of cycles without chords* (basic cycles).

A graph G is called *knot-homogeneous*, if each knot is embedded in the same (isomorph) set of cycles without chords. Obviously all knot-homogeneous graphs are regular. Knot-homogeneous ar for example all cycle-graphs C_n, complete graphs K_n, the graphs of the 5 platonic solids (tetrahedron, cube, octahedron, dodecahedron, icosahedron) and also the Petersen graph.

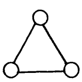
We characterize knot-homogeneous graphs by

$$G_{hom} := G(n,r,M);$$

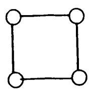
n is the number of knots, r the degree of regularity, and M is the set of all cycles whithout chords through some knot. If all cycles $c \in M$ have the same length, then you get a *perfect symmetry*.

The knowledge of M is not only useful for the discovering of symmetry or antisymmetry but also helpfull for regarding of coverings, decompositions, matroids etc.. Because there is a connection between basic cycles and contextures, it is possible to study the above marked hints from a new point of view.

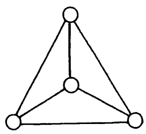
Be M_i the set of all cycles c through a certain knot k_i of a knot-homogeneous graph G_{hom}. From the property of knot-homogeneity follows: all M_i are isomorph. Also trivial is, that it is always possible to get a $B \subset M$ so, that B is sufficient for a complete covering of G_{hom} (basic set B). Some examples may illustrate the preceding remarks.



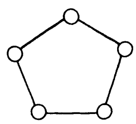
$G(3,2,C_3)$



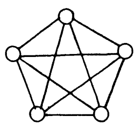
$G(4,2,C_4)$



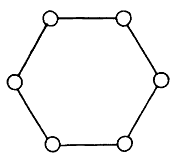
$G(4,3,3C_3)$



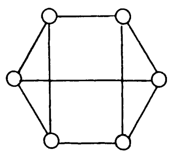
$G(5,2,C_5)$



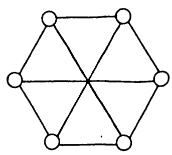
$G(5,4,6C_3)$



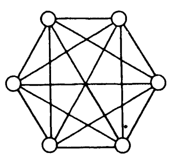
$G(6,2,C_6)$



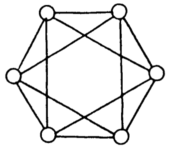
$G(6,3,C_3 2C_4)$



$G(6,3,6C_4)$



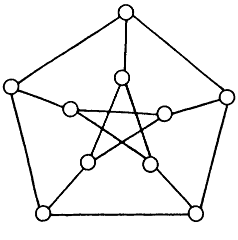
$G(6,5,10C_3)$



$G(6,4,4C_3 2C_6)$

The first 10 graphs are all knot-homogeneous graphs up to 6 knots.

Petersen graph →



$G(10,3,6C_5)$

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*Edges, which connect knots of a cycle, but do not belong to the cycle are called *chords*.

Supplement

The first two tables of the two balanced permutographs with a line-contexture resp. a star-contexture you can use to work with negations in a 4-valued negation-system. For example holds

in the negations-system with the line-contexture

$N_1N_2N_1N_3N_2N_1$ (1234)

$= N_1N_2N_1N_3N_2N_1$ ①

$= N_1N_2N_1N_3N_2$ ⑦

$= N_1N_2N_1N_3$ ⑬

$= N_1N_2N_1$ ⑲

$= N_1N_2$ ㉑

$= N_1$ ㉓ = ㉔ = (4321)

in the negation-system with the star-contexture

$N_1N_2N_1N_3N_2N_1$ (1234)

$= N_1N_2N_1N_3N_2N_1$ ①

$= N_1N_2N_1N_3N_2$ ⑦

$= N_1N_2N_1N_3$ ⑨

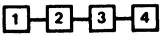
$= N_1N_2N_1$ ⑩

$= N_1N_2$ ④

$= N_1$ ⑭ = ⑯ = (3241)

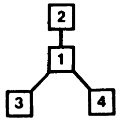
permutation		N1	N2	N3
1234	1	7	3	2
1243	2	8	4	1
1324	3	9	1	5
1342	4	10	2	6
1423	5	11	6	3
1432	6	12	5	4
2134	7	1	13	8
2143	8	2	14	7
2314	9	3	15	11
2341	10	4	16	12
2413	11	5	17	9
2431	12	6	18	10
3124	13	15	7	19
3142	14	16	8	20
3214	15	13	9	21
3241	16	14	10	22
3412	17	18	11	23
3421	18	17	12	24
4123	19	21	20	13
4132	20	22	19	14
4213	21	19	23	15
4231	22	20	24	16
4312	23	24	21	17
4321	24	23	22	18

Table of negations
Permutograph with line contexture



permutation		N1	N2	N3
1234	1	7	15	22
1243	2	8	16	21
1324	3	9	13	24
1342	4	10	14	23
1423	5	11	18	19
1432	6	12	17	20
2134	7	1	9	12
2143	8	2	10	11
2314	9	3	7	10
2341	10	4	8	9
2413	11	5	12	8
2431	12	6	11	7
3124	13	15	3	18
3142	14	16	4	17
3214	15	13	1	16
3241	16	14	2	15
3412	17	18	6	14
3421	18	17	5	13
4123	19	21	24	2
4132	20	22	23	6
4213	21	19	22	2
4231	22	20	21	1
4312	23	24	20	4
4321	24	23	19	3

Table of negations
Permutograph with star contexture



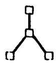
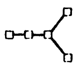
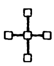
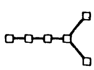
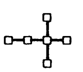
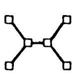
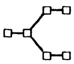
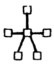
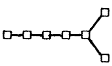
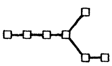
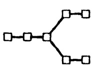
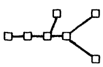
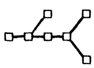
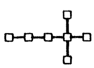
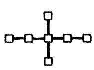
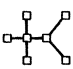
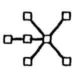
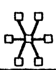
n	PG generating tree-contexture T(n)		Number of different basic cycles ■	⬢	Σ	Basic cycles (indices of negators) ■	⬢	Total number of basic cycles in PG ■	⬢	Σ
2		$T_1(2)$								
3		$T_1(3)$		1	1		121212		1	1
4		$T_1(4)$	1	2	3	1313	121212 232323	6	8	14
		$T_2(4)$	–	3	3		121212 131313 232323	–	12	12
5		$T_1(5)$	3	3	6	1313 1414 2424	121212 232323 343434	90	60	150
		$T_2(5)$	2	4	6	1313 1414	121212 232323 242424 343434	60	80	140
		$T_3(5)$	–	6	6		121212 131313 141414 232323 242424 343434	–	120	120
6		$T_1(6)$	6	4	10	1313 1414 1515 2424 2525 3535	121212 232323 343434 454545	1080	480	1560
		$T_2(6)$	5	5	10	1313 1414 1515 2424 2525	121212 232323 343434 353535 454545	900	600	1500
		$T_3(6)$	3	7	10	1313 1414 1515	121212 232323 242424 252525 343434 353535 454545	540	840	1380
		$T_4(6)$	4	6	10	1414 1515 2424 2525	121212 131313 232323 343434 353535 454545	720	720	1440

Table: Basic cycles in permutographs (with tree-contextures) on n-permutations

n	PG generating tree-contexture T(n)		Number of different basic cycles			Basic cycles (indices of negators)		Total number of basic cycles in PG		
			■	⬢	Σ	■	⬢	■	⬢	Σ
continue 6		$T_5(6)$	5	5	10	1414	121212	900	600	1500
						1515	131313			
						2525	232323			
						3434	242424			
						4545	353535			
		$T_6(6)$	–	10	10		121212			
							131313			
							141414			
							151515			
							232323			
							242424			
							252525			
							343434			
							353535			
							454545			
7		$T_1(7)$	10	5	15	In T(n) unneighboured edges form 4-cycles	In T(n) neighboured edges form 6-cycles	12600	4200	16800
		$T_2(7)$	9	6	15			11340	5040	16380
		$T_3(7)$	9	6	15			11340	5040	16380
		$T_4(7)$	9	6	15			11340	5040	16380
		$T_5(7)$	8	7	15			10080	5880	15960
		$T_6(7)$	8	7	15			10080	5880	15960
		$T_7(7)$	7	8	15			8820	6720	15540
		$T_8(7)$	7	8	15			8820	6720	15540
		$T_9(7)$	6	9	15			7560	7560	15120
		$T_{10}(7)$	4	11	15			5040	9240	14120
		$T_{11}(7)$	–	15	15			–	12600	12600

Acknowledgement

My thanks are due to Prof. G. Günther for his numerous helpful and agreeing remarks to work in the discussed direction and R. Kaehr [6], who impulsed me often to publish some of my ideas of the formalization of some points of the many-valued logic. And last, but not least I thank Mrs. H. Klein for her drawing of the graphic representations.

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