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In: Zdeněk Frolík (ed.): Proceedings of the 10th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1982. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 2. pp. [287]–292.

Persistent URL: http://dml.cz/dmlcz/701283

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ENTROPIC DIMENSION OF UNIFORM SPACES

Jiří Vilímovský

It will be shown that the concepts of the "entropic" dimension of compact metric spaces due to Pontrjagin and Schnirelmann [4] and the "approximative" dimension of locally convex topological vector spaces due to Pietsch [3] may be generalized to the case of a general uniform space. Comparison of this type of dimension with the covering dimension Δd will give the common generalization of several results obtained in [3] and [4].

Throughout the paper we will refer to [2] for basic definitions and results pertaining to uniform spaces and covering dimension. If A is a set, we shall denote H(A) the hedgehog over the set A, that is the set of all pairs $\langle a,x \rangle$, $a \in A$, $0 \leq x \leq 1$, where we consider $\langle a,0 \rangle = \langle b,0 \rangle$ for all $a,b \in A$ (the body of H(A)). The uniformity of H(A) will be defined by the metric

$$d(\langle a,x\rangle,\langle b,y\rangle) = \begin{cases} x + y & \text{if } a \neq b \\ |x - y| & \text{if } a = b \end{cases}$$

Recall that H(A) is a onedimensional injective uniform space [2], and that every finite-dimensional uniform cover of a uniform space may be realized by a uniformly continuous mapping into a sufficiently large product of hedgehogs [1].

At first we refine the last result in finding even n-dimensional spaces $T_n(A)$ such that every at most n-dimensional uniform cover of a uniform space may be realized in $T_n(A)$ for some sufficiently large set A.

<u>Definition 1</u>: Let A be a set, we define for all $n \in \omega$ the uniform subspace $T_n(A)$ of the product $H^{n+1}(A)$ of n+1 copies of H(A) as follows:

$T_o(A) = \{\langle a,1\rangle;\ a \in A\}$ with the uniformly discrete uniformity inherited by a cannonical embedding into H(A).

$T_n(A) = T_o(A) \times H^n(A) \cup H(A) \times T_o(A) \times H^{n-1}(A) \cup \ldots \cup H^n(A) \times T_o(A)$ for $n \geq 1$, the uniformity of it is induced by the cannonical embedding into $H^{n+1}(A)$.

So $T_n(A)$ is a subspace of $H^{n+1}(A)$ consisting of all points, at least one coordinate of which lies at the end of some spine of the hedgehog.

Immediately from the definition one may observe that the dimension $\Delta d T_n(A) = n$ for every n

Theorem 1 : Let X be a uniform space, then $\Delta dX \leq n$ if and only if for every uniform cover $\mathcal{U}$ of X there is a uniformly continuous mapping $f: X \to T_n(A)$ realizing $\mathcal{U}$, where A is any set of sufficiently large cardinality.

Proof: Take for A any set such that all uniformly discrete families in X have the cardinality at most $|A|$. X has a cover-basis consisting of covers of order at most n+1, therefore starting with any uniform cover $\mathcal{U}$ of X, we may find a uniform refinement $\mathcal{U}'$ of $\mathcal{U}$ of order at most n+1. We take $\mathcal{U}''$ a strict shrinking of $\mathcal{U}'$ and $\mathcal{V}$ a uniform (n+1)-discrete refinement of $\mathcal{U}''$. $\mathcal{V} = \bigcup \{\mathcal{V}_k;\ k=1,\dots,n+1\}$, where $\mathcal{V}_k$ are uniformly discrete families in X. We take the natural uniformly continuous mappings $f_k: \bigcup \mathcal{V}_k \to T_0(A)$. H(A) is an injective uniform space, hence every f_k may be extended to a uniformly continuous $\bar{f}_k: X \to H(A)$. Now we define for every $x \in X$ $f(x) = \{\bar{f}_k(x);\ k=1,\dots,n+1\}$. f is uniformly continuous into $H^{n+1}(A)$ and ranges in $T_n(A)$, because $\mathcal{V}$ covers X. From the construction now easily follows that $f^{-1}(\mathcal{W})$ refines $\mathcal{U}$ for a suitable uniform cover $\mathcal{W}$ of $T_n(A)$.

The sufficiency of the condition is evident.

Definition 2 : Let d be a pseudometric on a set X, $U \subset X$, $\varepsilon > 0$. We shall denote (cf. [3])

$$M_\varepsilon(U,d) = \sup\{m \in \omega;\ \exists x_1,\dots,x_m \in U \text{ such that } d(x_i,x_j) \geq \varepsilon \text{ for all } i \neq j\}$$

If U is precompact in d, the numbers $M_\varepsilon(U,d)$ are finite for all $\varepsilon > 0$ and the asymptotic behavior of the function M_ε of variable $\varepsilon > 0$ for $\varepsilon \to 0$ may serve as a measure of complexity of the space (U,d). We recall two (slightly modified) classical results mentioned in the foreword:

Theorem PS [4]: If (X,d) is a precompact metric space with $\Delta dX \geq n$, then there is a constant $c > 0$ such that for sufficiently small $\varepsilon > 0$ we have

$$M_\varepsilon(X,d) > \frac{c}{\varepsilon^n}$$

Theorem P [3]: If E is a locally convex topological vector space, the (algebraic) dimension of E is at most n if and only if for every continuous seminorm p we can find a neighborhood U of 0, such that

$$\overline{\lim_{\varepsilon\to 0}} \frac{M_\varepsilon(U,p')}{\varepsilon^{-n}} < \infty$$

where p'is the pseudometric related to the seminorm p.

Our aim is to generalize this sort of ideas and find a tool for measure of complexity of a uniform space using metrics and compactness even for spaces which are fare from being metrizable or (locally) precompact. At first we prove the following technical lemma:

Lemma 1 : Let $d_1,\dots,d_n$ be pseudometrics on sets $X_1,\dots,X_n$ respectively. Denote $\prod\{d_i(\{x_i\},\{y_i\});\ i=1,\dots,n\} = \max\{d_i(x_i,y_i);\ i\le n\}$. $\prod d_i$ is a pseudometric on $\prod X_i$. If $U_i\subset X_i$, then the following inequalities hold:

$$\prod M_\varepsilon(U_i,d_i) \le M_\varepsilon(\prod U_i,\prod d_i) \le \prod M_{\frac{\varepsilon}{2}}(U_i,d_i)$$

Proof: Suppose $D_i\subset U_i$ is ε-discrete in d_i, then $\prod D_i$ is ε-discrete in $\prod d_i$, therefore the first inequality is valid.
The second inequality holds evidently for n=1, because every ε-discrete subset is $\frac{\varepsilon}{2}$ - discrete. Assume the validity of the inequality for k and take the set $\prod\{U_i;\ i=1,\dots,k\}\times U_{k+1}$. Every $\frac{\varepsilon}{2}$ - maximal discrete subset $\{x_j;\ j=1,\dots,m\}$ of U_{k+1} is an $\frac{\varepsilon}{2}$ - net in U_{k+1}, that means that all open balls $B(x_j,\frac{\varepsilon}{2})$, $j=1,\dots,m$ in pseudometric d_{k+1} cover the set U_{k+1}.
Using our assumption we claim that for every $j = 1,\dots,m$ there are at most $\prod\{M_{\frac{\varepsilon}{2}}(U_i,d_i);\ i=1,\dots,k\}$ points forming an ε-discrete family in the set $\prod\{U_i;\ i=1,\dots,k\}\times B(x_j,\frac{\varepsilon}{2})$, hence every ε-discrete family in $\prod\{U_i;\ i=1,\dots,k+1\}$ has at most $m\cdot\prod\{M_{\frac{\varepsilon}{2}}(U_i,d_i);\ i=1,\dots,k\}$ points. The result follows now by induction.

Definition 3 : If $\mathcal{U}$ is a family of subsets of a set X, d is a pseudometric on X, $\varepsilon>0$, we define

$$M_\varepsilon(\mathcal{U},d) = \sup\{M_\varepsilon(U,d);\ U\in\mathcal{U}\}$$

Definition 4 : If X is a uniform space, the family $\mathcal{D}$ of uniformly continuous pseudometrics on X will be called a base for X, if for every uniform cover $\mathcal{U}$ of X there is a pseudometric $d\in\mathcal{D}$ such that $\mathcal{U}$ is a uniform cover of a pseudometric space (X,d).

Definition 5 : Let X be a uniform space. We shall say that the entropic dimension of X is at most n ($\mathrm{Ed}X\le n$), if there exists a basis $\mathcal{D}$ for uniformly continuous pseudometrics on X such that for every $d\in\mathcal{D}$ there is a uniform cover $\mathcal{U}$ of X such that the following is true:

$$\overline{\lim_{\varepsilon\to 0}} \frac{M_\varepsilon(\mathcal{U},d)}{\varepsilon^{-n}} < \infty$$

Proposition 1 : For any set A, $\mathrm{Ed}H(A) \leq 1$.

We notice that the usual metric d on H(A), though it is the most natural basis for uniformly continuous pseudometrics on H(A), cannot be used for our purposses, because no neighbourhood of the "body" is precompact in this metric. In spite of it it is possible to find another basis for uniformly continuous pseudometrics on H(A) having the desired properties.

Proof of Proposition 1 : For every $n \in N$ (natural numbers) define:

$$d_n(\langle a,r\rangle,\langle b,s\rangle) = \begin{cases} 0 \text{ if } r,s \leq \frac{1}{n} \\ r - \frac{1}{n} \text{ if } r > \frac{1}{n},\ s \leq \frac{1}{n} \\ s - \frac{1}{n} \text{ if } r \leq \frac{1}{n},\ s > \frac{1}{n} \\ |r - s| \text{ if } a=b,\ r,s > \frac{1}{n} \\ r + s - \frac{2}{n} \text{ if } a \neq b,\ r,s > \frac{1}{n} \end{cases}$$

Every d_n is a pseudometric on H(A) and the family $\mathcal{D} = \{d_n ; n \in N\}$ is a base for uniformly continuous pseudometrics on H(A). Take any $d_n \in \mathcal{D}$ and put $\mathcal{U}_n$ the uniform cover of H(A) consisting of all balls with radius $\frac{1}{2n}$ with respect to the usual metric d. Then evidently

$$M_\varepsilon(\mathcal{U}_n,d_n) \leq \frac{1}{\varepsilon} \quad \text{for all } \varepsilon > 0.$$

Now we are prepared to prove the main result:

Theorem 2 : Let X be a uniform space, if $\Delta dX \leq n$, then $\mathrm{Ed}X \leq n$.

Proof : Using Theorem 1 we can find a set S and a family $\{f_a;\ a \in J\}$ of uniformly continuous mappings from X into $T_n(S)$ such that for every uniform cover $\mathcal{U}$ of X, $f_a^{-1}(\mathcal{W})$ refines $\mathcal{U}$ for some $a \in J$ and some uniform cover $\mathcal{W}$ of $T_n(S)$.

Take $p \in \omega$, we shall take the following basis $\mathcal{D}$ for uniformly continuous pseudometrics on a product $H^p(S)$:

$\mathcal{D} = \{\prod\{d_i;i=1,\dots,p\} ;\ d_i \in \mathcal{D}_i\}$, where $\mathcal{D}_i$ are the bases on H(S) from Proposition 1.

Take any $d = \prod d_i \in \mathcal{D}$. For every i we take a uniform cover $\mathcal{U}_i$ of H(S) such that

$$M_\varepsilon(\mathcal{U}_i,d_i) \leq \frac{1}{\varepsilon} \quad \text{for every } \varepsilon > 0 .$$

Now take a uniform cover $\mathcal{U}$ of $H^p(S)$ defined:

$$\mathcal{U} = \{\prod\{U_i;\ i=1,\dots,p\};\ U_i \in \mathcal{U}_i\}.$$

Using Lemma 1 we may estimate

$$M_\varepsilon(\mathcal{U},d) \leq \prod\{M_{\frac{\varepsilon}{2}}(\mathcal{U}_i,d_i) ;\ i = 1,\dots,p\} \leq 2^p \varepsilon^{-p}$$

for all $\varepsilon > 0$. Furthermore $\mathcal{U}_i$ may be taken as fine, that their restriction to $T_o(S)$ consists of singletons only, hence for all $d_i \in \mathcal{D}_i$, $0 < \varepsilon < 1$, there is

$$M_\varepsilon(\mathcal{U}_i \upharpoonright T_o(S),\ d_i \upharpoonright T_o^2(S)\ ) = 1$$

If we put $p = n+1$, $\mathcal{D}' = \{d \upharpoonright T_n(S) \times T_n(S)\ ;\ d \in \mathcal{D}\}$, then $\mathcal{D}'$ is a basis for uniformly continuous pseudometrics on $T_n(S)$ and for every $d \in \mathcal{D}'$ we are able to find a uniform cover $\mathcal{U}$ of $T_n(S)$ such that for $0 < \varepsilon < 1$ there is

$$M_\varepsilon(\mathcal{U},d) \leq (n+1)\ 2^n\ \varepsilon^{-n}$$

All pseudometrics on X of the form $d \cdot f_a^2$, $a \in J$, $d \in \mathcal{D}'$ form a basis for uniformly continuous pseudometrics on X and we have

$$M_\varepsilon(\ f_a^{-1}(\mathcal{U}),\ d \cdot f_a^2\ ) \leq M_\varepsilon(\mathcal{U},d) \leq (n+1)\ 2^n\ \varepsilon^{-n}$$

for all $a \in J$ and all $0 < \varepsilon < 1$, hence

$$\overline{\lim_{\varepsilon \to 0}}\ \frac{M_\varepsilon(f_a^{-1}(\mathcal{U}),\ d \cdot f_a^2)}{\varepsilon^{-n}} \leqslant (n+1)\ 2^n < \infty$$

for all $a \in J$, and hence $\mathrm{Ed}X \leq n$.

We may define $\mathrm{Ed}X = n$ if $\mathrm{Ed}X \leq n$ and $\mathrm{Ed}X \nleq n-1$, and also $\mathrm{Ed}X = \infty$ if $\mathrm{Ed}X \nleq n$ for all natural numbers n. Under this notation we may reed the preceding theorem as $\mathrm{Ed}X \leq \Delta\mathrm{d}X$ for all uniform spaces X.

Unfortunately we do not know any example of a uniform space where Ed differs from Δd. At least for some spaces we are able to deduce that Ed coincides with Δd.

Proposition 2 : Let X be either precompact or uniformly embeddable onto a convex subset of a topological vector space. Then $\mathrm{Ed}X = \Delta\mathrm{d}X$.

Proof: If $\mathrm{Ed}X = \infty$, the statement follows from Theorem 2. Suppose that $\mathrm{Ed}X \leq n$.

If X is precompact, we can find a basis $\mathcal{D}$ for uniformly continuous pseudometrics on X such that for every $d \in \mathcal{D}$ there is a finite uniform cover $\mathcal{U}$ of X with

$$\overline{\lim_{\varepsilon \to 0}}\ \frac{M_\varepsilon(\mathcal{U},d)}{\varepsilon^{-n}} < \infty\ ,\ \text{hence}\ \overline{\lim_{\varepsilon \to 0}}\ \frac{M_\varepsilon(X,d)}{\varepsilon^{-n}} < \infty\ .$$

Using Theorem PS we get that d is at most n-dimensional, therefore $\Delta\mathrm{d}X \leq n$.

The second case follows from the first one, because each at least n-dimensional convex subset of a topological vector space contains a precompact subspace of dimension at least n.

Concluding remarks: The entropic measure of complexity of a uniform space may be used to classify also more complicated spaces than the finite-dimensional ones only. If one defines for a uniform space X with basis $\mathcal{D}$ for uniformly continuous pseudometrics

$$\mathcal{E}(X,\mathcal{D}) = \left\{\varphi : (0,1) \to (0,\infty)\ ;\ \text{for every } d \in \mathcal{D} \text{ there is a uniform cover } \mathcal{U} \text{ of } X \text{ with } \overline{\lim_{\varepsilon \to 0}}\ \frac{1}{\varphi(\varepsilon)}\, M_\varepsilon(\mathcal{U},d) < \infty\right\},$$

and for a uniform space X

$$\mathcal{E}(X) = \bigcup \{ \mathcal{E}(X, \mathcal{D}) \ ; \ \mathcal{D} \text{ is a basis for uniformly continuous pseudometrics on } X \},$$

then $\mathcal{E}(X) \neq \emptyset$ also for many infinite-dimensional uniform spaces and may serve as the dimension-like classification of X. For example it follows directly from a result of Pietsch [3], that if X is a nuclear locally convex topological vector space with its natural uniformity, then $2^{\varepsilon^{-1}}$ belongs to $\mathcal{E}(X)$.

Using this notation the main results of this paper reed as follows:

If ΔdX is at most n, then ε^{-n} belongs to $\mathcal{E}(X)$.

If X is either precompact or a convex subset of a topological vector space, then $\Delta dX \leq n$ if and only if $\varepsilon^{-n} \in \mathcal{E}(X)$.

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