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SIMULTANEOUS STRATEGIES AND BOOLEAN GAMES OF UNCOUNTABLE LENGTH

Peter Vojtáš

The paper is devoted to the study of the existence of a λ -closed dense subsets of a Boolean algebra under certain game-theoretical properties. T.Jech in [3] introduced the following game. Let B be a complete Boolean algebra and α an ordinal number. The transfinite game $\mathcal{G}(B, \alpha)$ is played between two players White and Black. Let White and Black define a decreasing sequence

$$w_0 \geq b_1 \geq w_2 \geq b_3 \geq \dots \geq w_{\beta+2n} \geq b_{\beta+2n+1} \geq \dots \quad (1)$$

of nonzero elements of B of the length $\leq \alpha$ by taking turns defining its entries. (White chooses $w_{\beta+2n}$, for β limit $< \alpha$, $n \in \omega$; Black chooses $b_{\beta+2n+1}$) The play is won by Black if the sequence (1) has nonzero intersection and length α , and by White if the intersection is \emptyset . A winning strategy for Black in the game $\mathcal{G}(B, \alpha)$ is a function $\zeta : \bigcup \{ {}^\beta B : \beta < \alpha \} \rightarrow B$ with the property that Black wins every play (1) in which he follows ζ .

Theorem 1 (T.Jech [3]). Assume B is a complete Boolean algebra and λ an uncountable cardinal number. Then $(a) \rightarrow (c) \rightarrow (d)$, where

- (a) Algebra B has a λ -closed dense subset.
- (c) Black has a winning strategy in the game $\mathcal{G}(B, \tau)$ for each $\tau < \lambda$.
- (d) Algebra B is $(\tau, \infty, 2)$ distributive for each $\tau < \lambda$.

Basically, our research was motivated by the question whether $(c) \rightarrow (a)$, i.e. whether the existence of winning strategy for Black implies that B has a λ -closed dense subset. The following definition concerns the structure of the set of all strategies and is powerfull also for limit cardinal numbers.

Definition 2. We say that Black has a simultaneous winning strategy of the length λ in the algebra B if there is one strategy $\zeta : \bigcup \{ {}^\alpha B : \alpha < \lambda \} \rightarrow B$ such that ζ is winning for Black in each game $\mathcal{G}(B, \alpha)$ for $\alpha < \lambda$.

Lemma 3. $(a) \rightarrow (b) \rightarrow (c)$, where

- (a) and (c) are as in the Theorem 1, and

(b) Black has simultaneous strategy for B of the length λ .

PROOF. Is obvious.

In what follows we describe a type of algebras for which implication (b) \rightarrow (a) holds, namely the ones which has a tree-base. We give some conditions under which a boolean algebra has a tree-base. As a consequence we get an characterization of the algebra $\text{Col}(\lambda, \kappa)$. At the end of this paper we give some historical comments.

§1. Notations, definitions, constructions. Let B be a complete Boolean algebra, \leq is the canonical ordering of the algebra B , λ, κ, τ are cardinal and α, β, γ ordinal numbers, Lim denotes the class of limit ordinal numbers. $B^+ = B - \{0\}$, $B \upharpoonright u$ is a partial algebra, $\text{hsat}(B)$ denotes the hereditarily saturatedness. By P and Q we denote a maximal partition of B , system $\mathcal{M} = \{P_\alpha : \alpha < \lambda\}$ is called a matrix, P_α 's are columns of \mathcal{M} , $x \in P_\alpha$ is an element of the matrix \mathcal{M} , $P \ll Q$ denotes that P refines Q , $P \ll \mathcal{M}$ if P refines each P_α . \mathcal{M} is said to be monotone provided $\alpha < \beta$ implies $P_\beta \ll P_\alpha$. Remark, that if \mathcal{M} is monotone, then $(\bigcup \mathcal{M}, \leq)$ forms a tree and P_α is the α -th level of this tree. Let $\Omega = \{Q_\alpha : \alpha < \lambda\}$. Then Ω refines \mathcal{M} if each Q_α refines P_α . For $x \in B^+$, $x \wedge \wedge P = \{y \wedge x : y \in P_\alpha\} \cap B^+$. The algebra B is said to be $(\lambda, \infty, \kappa)$ -distributive provided for any matrix $\mathcal{M} = \{P_\alpha : \alpha < \lambda\}$ there is a maximal partition P of B such that $(\forall x \in P)(\forall \alpha < \lambda)(|x \wedge \wedge P_\alpha| < \kappa)$. Algebra B is called $(\lambda, \infty, \kappa)$ -nowhere distributive if for each $x \in B^+$ the algebra $B \upharpoonright x$ is not $(\lambda, \infty, \kappa)$ -distributive. Recall that B is $(\lambda, \infty, \kappa)$ -nowhere distributive iff there is a matrix $\mathcal{M} = \{P_\alpha : \alpha < \lambda\}$ such that for each $x \in B^+$ there is some $\alpha < \lambda$ with $|x \wedge \wedge P_\alpha| \geq \kappa$. In this case we say that \mathcal{M} is a matrix witnessing to $(\lambda, \infty, \kappa)$ -nowhere distributivity and if B is $(\tau, \infty, 2)$ -distributive for all $\tau < \lambda$ (e.g. if B has simultaneous winning strategy for Black of length λ) then \mathcal{M} will be assumed to be monotone. We say that $D \subseteq B^+$ is a λ -closed dense subset of algebra B (we say sometimes base instead of dense subset) if $(\forall x \in B^+)(\exists y \in D)(y \leq x)$ and for every decreasing sequence $\{a_\alpha : \alpha < \tau\} \subseteq D$ of the length $\tau < \lambda$ there is an $y \in D$ such that $y \leq a_\alpha$ for each $\alpha < \tau$. $d(B)$ denotes the density of B . A matrix $\mathcal{M} = \{P_\alpha : \alpha < \lambda\}$ is said to be a tree-base of the algebra B of the length λ if $\bigcup \mathcal{M}$ is a base (i.e. elements of \mathcal{M} form a base) and \mathcal{M} is monotone (i.e. $(\bigcup \mathcal{M}, \leq)$ is a tree). For unexplained notation we refer to [2].

§2. Tree-base, game-tree and simultaneous strategy. The idea to construct a λ -closed dense subset of a Boolean algebra from a

tree appeared independently in [1] and [4]. We develop the techniques of [4] to obtain more general results.

Theorem 4. Assume B is a complete Boolean algebra which has a tree-base of the length λ and B has simultaneous winning strategy for Black of length λ . Then B has a λ -closed dense subset.

PROOF. The idea of the proof is following. We introduce a natural notion of a game tree according to strategy σ for Black in which every branch is a play of game \mathcal{G} in which Black follows σ and the elements of the tree split on even levels (turns of the White) and odd levels are determined by even ones and σ . Then by transfinite induction we construct a game-tree according to σ which refines the base \mathcal{H} and this is the desired λ -closed base.

Let $\mathcal{H} = \{P_\alpha : \alpha < \lambda\}$ be a tree-base of B and σ is a simultaneous winning strategy. A tree (T, \leq) is called a game-tree of length λ according to the strategy σ if

- (i) $T \subseteq B^+$, \leq is the canonical ordering of B , T has length λ
- (ii) $(\forall \alpha \in \lambda \cap \text{Lim})(\forall n \in \omega) T_{\alpha+2n+1}$ is a maximal partition of B (where T_β denotes the β -th level of the tree T)
- (iii) $(\forall x \in T_{\alpha+2n+1})(x = \sigma(\text{pr}(x)))$ (where $\text{pr}(x)$ is the sequence of predecessors of x in the tree T).

Note that any branch of T is a play of game \mathcal{G} in which Black follows σ , so it is of the length λ . So if T is a base of algebra B it is a λ -closed base. To this end we have to construct a game-tree T which refines the tree base \mathcal{H} i.e.

- (iv) $(\forall \alpha \in \lambda \cap \text{Lim})(\forall n \in \omega)(T_{\alpha+2n+1} \ll P_{\alpha+n})$

Assume that $T_{\alpha+2n}$ is constructed already. For $T_{\alpha+2n+1}$ take the maximal system such that

- (v) $T_{\alpha+2n}$ refines $T_{\alpha+2n+1}$
- (vi) $\{\sigma(\text{pr}(x), x) : x \in T_{\alpha+2n+1}\}$ is a maximal partition of B

which refines $P_{\alpha+n}$.

This is possible because $T_{\alpha+2n}$ is a game tree of the length $\lambda+2n$ according to the simultaneous strategy σ of the length $\lambda+2n$, algebra B is complete and $(\lambda, \infty, 2)$ distributive.

q.e.d.

Now we consider the question, when does there exists a tree-base for a Boolean algebra. We need the following technical lemma.

Lemma 5. Assume λ is an uncountable cardinal number, B is a complete Boolean algebra which has a simultaneous winning strategy for Black of length λ and $\mathcal{H} = \{P_\alpha : \alpha < \lambda\}$ is a monotone matrix witnessing to $(\lambda, \infty, \kappa)$ -nowhere distributivity of B . Then for each $x \in B^+$ the following holds

(i) $|\{(y, \alpha) : y \in P_\alpha \text{ \& } x \wedge y \neq \emptyset\}| \geq \kappa^\lambda$ (i.e. x intersects many elements of the matrix \mathcal{M})

and (ii) $\text{hsat}(B) > \kappa^\lambda$

PROOF. The main idea again is to construct something like a game tree of type $(\lambda, \kappa, \subseteq)$, but its relation to the matrix \mathcal{M} is a little bit more complicated as in the Theorem 4.

Assume w.l.o.g. that $x = 1$. In fact for (i) we have to prove that $|\bigcup \mathcal{M}| \geq \kappa^\lambda$. The existence of simultaneous strategy of length λ implies $(\mathfrak{T}, \infty, 2)$ -distributivity for $\mathfrak{T} < \lambda$ and B is $(\lambda, \infty, \kappa)$ -nowhere distributive - we have that λ is regular. By transfinite induction we construct F and T such that

(i) $T \subseteq \bigcup \mathcal{M}$ is a Boolean tree (i.e. (i) of Theorem 4 holds)

(ii) $F : \bigcup \{\alpha : \alpha < \lambda\} \rightarrow T$ is an tree isomorphism

(iii) for each $f \in {}^\lambda \kappa$ the sequence $F(f \upharpoonright 0) = w_0, \mathfrak{G}(w_0) = b_1, w_2 = b_1 \wedge F(f \upharpoonright 1), b_3 = \mathfrak{G}(w_2), \dots, w_{\alpha+2n} = \bigwedge \{b_\beta : \beta < \alpha + 2n\} \wedge F(f \upharpoonright \alpha + 2n), b_{\alpha+2n+1} = \mathfrak{G}(w_{\alpha+2n}), \dots$ is a play in which Black follows \mathfrak{G} - the simultaneous strategy of the length λ .

As $T \subseteq \bigcup \mathcal{M}$ and \mathcal{M} is monotone and λ is regular we have immediately (i). For $\alpha < \lambda$ T_α is a disjoint system, so $\text{hsat}(B) \geq \kappa^\lambda$. If $\kappa^\lambda > \lambda$ then κ^λ is a singular cardinal number and so $\text{hsat}(B) > \kappa^\lambda$. If $\kappa^\lambda = \lambda$ then using \mathfrak{G} we can construct a strictly decreasing tower $\{t_\alpha : \alpha < \lambda\}$ and then $\{t_\alpha - t_{\alpha+1} : \alpha < \lambda\}$ is the desired disjoint system of power $\kappa^\lambda = \lambda$.

q.e.d.

Theorem 6. Let B be a $(\lambda, \infty, \kappa)$ -nowhere distributive complete Boolean algebra with density $d(B) = \kappa^\lambda$ for which the Black has simultaneous winning strategy of the length λ . Then B has a tree base of length λ .

PROOF. Let $D \subseteq B^+$ be a base of, size κ^λ . As each element of D intersects κ^λ elements of $\Omega = \{Q_\alpha : \alpha < \lambda\}$ - a monotone matrix witnessing to $(\lambda, \infty, \kappa)$ -nowhere distributivity - there is a one to one mapping $f : D \rightarrow \bigcup \Omega$ such that $x \wedge f(x) \neq \emptyset$. Split each $y \in \text{rng}(f)$ into two elements $y \wedge f^{-1}(y)$ and $y - f^{-1}(y)$. We obtain a matrix Ω' which refines D , i.e. $\bigcup \Omega'$ is a base. Using $(\mathfrak{T}, \infty, 2)$ distributivity for each $\mathfrak{T} < \lambda$ we obtain a monotone matrix \mathcal{M} - the desired tree-base.

q.e.d.

Corollary 7. Let B be a $(\lambda, \infty, \kappa)$ -nowhere distributive complete Boolean algebra with density $d(B) = \kappa^\lambda$ which has a simultaneous winning strategy for Black of length λ . Then

$$B \cong \text{Col}(\lambda, \kappa^\lambda) = \text{Col}(\lambda, \kappa).$$

PROOF. From Lemma 5 we have $\text{hsat}(B) = (\kappa^\delta)^+$ so in the construction of the Theorem 4 we can add the condition

$$(\forall x \in T_{\alpha+2n+1})(|\{y \in T_{\alpha+2n+2} : y \leq x\}| \geq \kappa^\delta).$$

The base we obtain is isomorphic to the base of the algebra $\text{Col}(\lambda, \kappa^\delta)$.

q.e.d.

We have seen that the existence of a tree base is important. Observe that if B has a tree base of the length λ and $\kappa < \text{hsat}(B)$ then B is $(\lambda, \infty, \kappa)$ -nowhere distributive. About inverse implication we know only what Theorem 6 says.

The case $\lambda = \omega_1$ was studied in [4]. This paper is a generalization of the results of [4] for $\lambda = \omega_1$. For nonlimit cardinal numbers M , Foreman in [1] proved that if $\kappa = \lambda^+$, $\lambda^\delta = \lambda$, B is $(\lambda, \infty, 2)$ -distributive, $d(B) = \lambda^+$ and Black wins $\mathcal{G}(B, \gamma)$ then B has a γ^+ -closed dense subset. Our technics works without cardinal assumptions and also for limit cardinals, but needs stronger strategical assumption.

The notion of the simultaneous strategy gives a view into the structure of all strategies. The problem, whether (c) \rightarrow (a), ((a), (b), (c) are as in Theorem 1 and Lemma 2) can be split into two: whether (c) \rightarrow (b) and (b) \rightarrow (a).

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