Ehrhard Behrends

M-Structure (a survey)

In: Zdeněk Frolík (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 3. pp. [15]–21.

Persistent URL: http://dml.cz/dmlcz/701290

M - STRUCTURE (A SURVEY)

Ehrhard Behrends

Abstract: In this talk the basic definitions, some fundamental theorems as well as some directions of applications of M-structure theory have been presented.

1. The basic definitions

The aim of M-structure theory is, roughly speaking, to describe how a given Banach space behaves like a space of continuous functions. This is done by defining objects (subspaces, operators) in arbitrary Banach spaces which in the case of CK-spaces are suitable to characterize these spaces. The fundamental definitions of the theory are due to Cunningham [16] and Alfsen-Effros [1,2].

Definition: Let X be a real Banach space, $J \subset X$ a closed subspace

(i) J is called an M-summand (resp. L-summand) if there is a closed subspace $J^{\perp}$ such that $X = J \oplus J^{\perp}$ algebraically and $\|x + x^{\perp}\| = \max\{\|x\|, \|x^{\perp}\|\}$ (resp. $\|x+x^{\perp}\| = \|x\| + \|x^{\perp}\|$) for $x \in J$, $x^{\perp} \in J^{\perp}$.

(ii) J is called an M-ideal if J^{π}, the annihilator of J in X', is an L-summand.

(iii) An operator $T: X \to X$ is called a multiplier if for every extreme functional p on X there is an $a_T(p) \in \mathbb{R}$ such that $p \circ T = a_T(p)p$. $Z(X)$ (the centralizer of X) denotes the collection of all multipliers.

Examples:

1. $\{o\}$ and X are always L-summands (M-summands), $Z(X)$ always contains $\mathbb{R}\,\mathrm{Id}$.
2. If $X = CK$, then
   - X contains only $\{o\}$ and X as L-summands
   - the M-summands of X are precisely the subspaces $\{f \mid f \in CK, f|_A = o\} =: J_A$, where $A \subset K$ clopen

- the M-ideals of X are precisely the subspaces J_A, where $A \subset K$ closed
- $Z(X)$ contains just the operators $M_h : f \mapsto hf$, where h runs through X.

3. Let X be the self-adjoint part of a W*-algebra A. Then the M-ideals of X are just the self-adjoint parts of the closed two-sided ideals in A, and the operators in $Z(X)$ are precisely the operators $a \mapsto za$ ($z = z^*$, z in the center of A). In particular, $[K(H)]_{sa}$ is an M-ideal in $[B(H)]_{sa}$ (H Hilbert space); this has been the first interesting M-ideal in the literature (Dixmier [20]).

2. Some fundamental theorems

Since the beginning of M-structure theory many authors have contributed to this field. Most of the results are concerned with special aspects, and there is no hope to give a survey here (see, however, section 3). There are some frequently used theorems which apply to arbitrary situations, the most important of them (in the author's opinion) are the following:

- The characterization theorem for M-ideals
  Alfsen and Effros [1,2] have shown that it is possible to characterize M-ideals without using the dual space by means of an intersection property:

  A closed subspace J of X is an M-ideal iff
  $J \cap B_1 \cap B_2 \cap B_3 \neq \emptyset$ for every collection of three open balls B_1, B_2, B_3 such that $B_1 \cap B_2 \cap B_3 \neq \emptyset$,
  $B_i \cap J \neq \emptyset$ $(i=1,2,3)$.
- The abstract Dauns-Hofmann theorem
  This theorem (which is also due to Alfsen and Effros; cf. also [21]) relates the notions of multipliers and M-ideals. It states that the multipliers correspond to those bounded functions on the extreme boundary of the dual unit ball which are continuous with respect to a topology defined by means of the M-ideals of the space.
- The function module representation theorem. Cunningham [16] has shown that the above example **2**, where we described the multiplier of CK-spaces, is typical in the following sense:
  every X can be regarded as a space of vectorvalued functions over a compact Hausdorff space K such that the $T \in Z(X)$ correspond to the multiplication operators associated with the

continuous functions on K .

The L-M-theorem

This theorem (due to the author [3]) states the surprising fact that a space X cannot have nontrivial (i.e. different from $\{o\}$ and X) L-summands and M-summands at the same time.

More generally: If one extends the definition of L-summands and M-summands to those of L^p-summands (where the relevant norm condition is $\|x + x^{\perp}\|^p = \|x\|^p + \|x^{\perp}\|^p$) then for at most one p there can exist non-trivial L^p-summands.

3. Some applications of M-structure theory

I. Approximation theory

On the one hand, M-ideals have interesting approximation theoretical properties (they are proximal in a very strong sense). On the other hand, $K(H)$ is an M-ideal in $B(H)$, and these two facts motivated a number of authors ([15,29,30,31,32]) to approximate operators on a Hilbert space by compact operators. In order to have similar techniques for more Banach spaces the following problem has been of interest: For what Banach spaces X is it true that $K(X)$ is an M-ideal in $B(X)$. This problem is far from being solved, partial answers can be found in [24,26,27,35,36,45].

One can regard the nice behaviour of $K(H)$ in $B(H)$ also under another viewpoint. Since $B(H)$ is the bidual of $K(H)$ it is interesting to investigate those spaces X such that X is an M-ideal in its bidual. Spaces with this property have been treated in [25,37].

II. M-Structure and L^1-preduals

CK-spaces are the most simple examples of spaces for which the dual is an L^1-space. These spaces have been studied intensively during the last decade, and it is not surprising that M-structure plays a rôle in this theory. Using M-structure methods several authors have been able to give new characterizations for known classes of L^1-preduals ([22,38,39,40]) or to define and investigate interesting new classes ([17,41,42]).

Recently the author [9] has obtained a theorem that states that all L^1-preduals share a certain symmetry property, and the formulation and the proof depend heavily on M-structure methods.

III. Vector-valued Banach-Stone theorems

A Banach space X is called to have the Banach-Stone property if $C(M,X) \cong C(N,X)$ always implies that M and N are homeomorphic (for compact Hausdorff spaces M,N). Clearly $\mathbb{R}$ has the Banach-Stone property by the classical Banach-Stone theorem. Vector-valued Banach-Stone theorems are due to Jerison, Cambern and Sundaresan ([10-14,33,47]). The author has shown that the most far-reaching results can be obtained by using M-structure theory. To be more precise:

III_1: Suppose that $Z(X) = \mathbb{R}\,Id$. Then X has the Banach-Stone property [5]
(already this result contains most of the Banach-Stone theorems which have been obtained without using M-structure theory).

III_2: Suppose that X is a Banach space such that $Z(X)$ is finite-dimensional. Then X can uniquely be written as $X = \prod_{i=1}^{k} X_i^{r_i}$, where the X_i are pairwise not isometrically isomorphic and have one-dimensional centralizer.
X has the Banach-Stone property iff $\min_{i=1,\dots k} r_i = 1$ [4].
(this generalizes all other Banach-Stone theorems).

III_3: There is a method (described in [6,7,8] by which one can obtain Banach-Stone theorems involved one wants to. The precise formulation (which de… ls on the function module representation theorem) is som… at lengthy and is therefore omitted here.

III_4: Finally, it has recently be shown in [8] that in a sense every "reasonable" vector-valued Banach-Stone theorem can be proved by using M-structure methods.

REFERENCES

[1] E.M. Alfsen-E.G. Effros: Structure in real Banach spaces I Ann. of Math. 96, 1972, 98-128

[2] E.M. Alfsen-E.G. Effros: Structure in real Banach spaces II Ann. of Math. 96, 1972, 129-173

[3] E. Behrends: L^p-Struktur in Banachräumen, Studia Math. 55, 1976, 71-85

[4] E. Behrends: On the Banach-Stone theorem, Math. Annalen 233, 1978, 261-272

[5] E. Behrends: An application of M-structure to theorems of the Banach-Stone type, in: Notas de Matemâtica, North Holland Math. Studies 27 (Proceedings on the Paderborn Conference on Functional Analysis 1976), 1977, 29-49

[6] E. Behrends-U. Schmidt-Bichler: M-structure and the Banach-Stone theorem, Studia Math. 69, 1980, 33-40

[7] E. Behrends: M-structure and the Banach-Stone-Theorem. Lecture Notes in Math. 736, Springer-Verlag Berlin, 1979

[8] E. Behrends: How to obtain vector-valued Banach-Stone theorems by using M-Structure methods Math. Ann. 261, 1982, 387-398

[9] E. Behrends: Normal operators and multipliers on complex Banach spaces and a symmetry property of L^1-predual spaces. Preprint

[10] M. Cambern: A generalized Banach-Stone theorem, Proc. of the Am. Math. Soc. 17, 1966, 396-400

[11] M. Cambern:On mappings of spaces of functions with values in a Banach space, Duke Math. J. 42, 1975, 91-98

[12] M. Cambern: Isomorphisms of spaces of continuous vector-valued functions, Ill. J. of Math. 20, 1976, 1-11

[13] M. Cambern: The Banach-Stone property and the weak Banach-Stone property in three-dimensional spaces, Proc. of the Am. Math. Soc. 67, 1977, 55-61

[14] M. Cambern: Reflexive spaces with the Banach-Stone property Revue Roumaine de Math. pures et appl. (to appear)

[15] C.K. Chui et al.: L-ideals and numerical range preservation Ill. J. of Math. 21, 1977, 365-373

[16] F. Cunningham: M-structure in Banach spaces, Proc. of the Cambr. Phil. Soc. 63, 1967, 613-629

[17] F. Cunningham: Square Banach spaces, Proc. of the Cambr. Phil. Soc. 66, 1969, 553-558

[18] F. Cunningham-E.G. Effros-N.M. Roy: M-structure in dual Banach spaces, Israel J. of Math. 14, 1973, 304-308

[19] F. Cunningham-N.M. Roy: Extreme functionals on an upper semi-continuous function space, Proc. of the Am. Math. Soc. 42, 1974, 461-465

[20] J. Dixmier: Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert, Ann. of Math. 51, 1950, 387-408

[21] G.A. Elliott-D. Olesen: A simple proof of the Dauns-Hofmann theorem, Math. Scand. 34, 1974, 231-234

[22] A.J. Ellis et al.: Facial characterizations of complex Lindenstrauss spaces,Trans. of the A M S 268, 1981, 173-186

[23] R. Evans: A characterization of M-summands, Proc. of the Cambr. Phil. Soc. 76, 1974, 157-159

[24] H. Fakhoury: Sur les M-ideaux dans les espaces d'opérateurs (Preprint)

[25] P. Harmand-A. Lima: Banach spaces which are M-ideals in their bidual, to appear in: Trans. of the A M S

[26] J. Hennefeld: A decomposition of B(X)* and unique Hahn-Banach extensions. Pac. J. 46, 1973 , 197-199

[27] J. Hennefeld: M-ideals, HB-subspaces, and compact operators, Indiana Univ. Math. J. 28, 1979 , 927-934

[28] B. Hirsberg: M-ideals in complex function spaces and algebras Israel J. of Math. 12, 1972, 133-146

[29] R. Holmes: M-ideals in approximation theory, in: Approximation theory II, Academic Press 1976, 391-396

[30] R. Holmes-B. Kripke: Best approximation by compact operators Indiana Univ. Math. J. 21, 1971, 255-263

[31] R. Holmes-B. Scranton-J. Ward: Approximation from the space of compact operators and other M-ideals, Duke Maht. J. 42, 1975, 259-269

[32] R. Holmes-B. Scranton-J. Ward: Best approximation by compact operators II, Bull. of the Am. Math. Soc. 80, 1974, 98-102

[33] M. Jerison: The space of bounded maps into a Banach space, Ann. of Math. 52, 1950, 309-327

[34] Å. Lima: Intersection properteis of balls and subspaces in Banach spaces, Trans. of the Am. Math. Soc. 227, 1977, 1-62

[35] Å. Lima: Intersection properties of balls in spaces of compact operators. Ann. Inst. Fourier 28, 1978 , 35-65

[36] Å. Lima: M-ideals of compact operators in classical Banach spaces. Math. Scand. 44, 1979 , 207-217

[37] Å. Lima: On M-ideals and best approximation. Indiana Univ. Math. J. 31, 1982 , 27-36

[38] Å. Lima et al.: Intersections of M-ideals and G-spaces (Preprint)

[39] T.S.S.R.K. Rao: Characterizations of some classes of L^1-Preduals by the Alfsen-Effros structure topology, Israel J. of Math. 42, 1982, 20-32

[40] A.K. Roy: Convex functions on the dual ball of a complex Lindenstrauss space, J. London Math. Soc. (2) 20, 1979, 529-540

[41] N.M. Roy: A characterization of square Banach spaces, Israel J. of Math. 17, 1974, 142-148

[42] N.M. Roy: Contractive projections in square Banach-spaces Proc. of the Am. Math. Soc. 59, 1976, 291-296

[43] K. Saatkamp: M-ideals of compact operators. Math. Z. 158, 1978 , 253-263

[44] R.R. Smith, J.D. Ward: M-ideal structure in Banach algebras, J. Func. Anal. 27, 1978 , 337-349

[45] R.R. Smith, J.D. Ward: M-ideals in $B(l^p)$. Pac. J. 81, 1979 227-237

[46] R.R. Smith, J.D. Ward: Applications of convexity and M-ideal theory to Quotient Banach Algebras. Quart. J. Math. Oxford 30, 1979 , 365-384

[47] K. Sundaresan: Spaces of continuous functions into a Banach space, Studia Math. 48, 1973, 15-22

Ehrhard Behrends
Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6

D-1ooo Berlin 33