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SOME INTEGRAL FORMULAS IN COMPLEX CLIFFORD ANALYSIS .

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ABSTRACT.

In the paper a generalization to the case of complex Clifford analysis of methods used by V.Souček in [1] is presented. It is shown that the complexified integral formula of M.Riesz for solutions of complex Laplace equation, given in [2] follows from an integral formula in complex Clifford analysis. The details of proofs are omitted here. The author is indepted to V.Souček for the suggestion of the problem and valuable advices and discussions.

1. COMPLEX CLIFFORD ANALYSIS .

Complex Clifford analysis was studied by J.Ryan in the serie of papers (see e.g. [3]). I want to present here some basic facts of the theory and to prove an integral formula.

Let $\mathcal{C}_m^{\mathbb{C}} = \mathcal{C}_m \otimes \mathbb{C}$ be a complex Clifford algebra of an odd dimensional space $\mathbb{C}_m$ and let $m = n+1 = 2h \geqq 4$.

Take a basis $\{e_1, \dots e_n\}$ of the algebra $\mathcal{C}_m^{\mathbb{C}}$, suppose that

$$e_i^2 = -e_o \quad , i = 1,\dots,n$$
$$e_i e_j + e_j e_i = 0 \quad , i \neq j, i,j = 1,\dots n$$

holds, where e_o is the identity of $\mathcal{C}_m^{\mathbb{C}}$.

In the natural grading of $\mathcal{C}_m^{\mathbb{C}}$ we make identifications

$$(\mathcal{C}_m^{\mathbb{C}})_0 \equiv \mathbb{C} \quad , (\mathcal{C}_m^{\mathbb{C}})_1 \equiv \mathbb{C}_m \quad , (\mathcal{C}_m^{\mathbb{C}})_0 \oplus (\mathcal{C}_m^{\mathbb{C}})_1 = \mathbb{C}_{m+1}$$

For $z \in \mathbb{C}_{m+1}$, $z = \sum_{\alpha=0}^{m} z_\alpha e_\alpha$ we put $z^+ = z_o e_o - \sum_{j=1}^{m} z_j e_j$ and $\|z\| = z z^+ = \sum_{\alpha=0} (z_\alpha)^2$.

There are two special real subspaces of $\mathbb{C}_{m+1}$, namely Minkowski space

$$\mathcal{M}_{m+1} = \left\{ z = \sum_{\alpha=0}^{m} z_\alpha e_\alpha \;\middle|\; z_o = \bar{z}_o \quad , z_j = -\bar{z}_j, j=1,\dots n \right\}$$

and (negative) Euclidean space

$$\mathcal{E}_{m+1} = \left\{ z = \sum_{\alpha=0}^{m} z_\alpha e_\alpha \;\middle|\; z_\alpha = -\bar{z}_\alpha \;, \alpha = 0,1,\dots,n \right\}$$

The whole basis of $\mathcal{C}_m^{\mathbb{C}}$ as a complex vector space is

$$\left\{ e_A = e_{i_1} \dots e_{i_r} \;\middle|\; A = (i_1,\dots,i_r) \;,\; 1 \leqq i_1 < \dots < i_r \leqq n \right\}$$

Let $\mathcal{W} \subset \mathbb{C}_{m+1}$ be an open subset, denote $\mathcal{A}(\mathcal{W})$ the set of all mappings $f: \mathcal{W} \to \mathcal{C}_m^{\mathbb{C}}$ which are holomorphic on $\mathcal{W}$ i.e. if $f = \sum_A f_A e_A$ then the functions f_A are holomorphic for every multiindex A.

There are two differential operators

$$\partial = \sum_{\alpha=0}^{m} e_\alpha \frac{\partial}{\partial z_\alpha} \qquad , \quad \partial^+ = e_0 \frac{\partial}{\partial z_0} - \sum_{j=1}^{m} e_j \frac{\partial}{\partial z_j}$$

acting on function from $\mathcal{A}(\mathcal{W})$ from the right or from the left so we have for every $f \in \mathcal{A}(\mathcal{W})$ the following elements of $\mathcal{A}(\mathcal{W})$

$$(f\partial) \;,\; (\partial f) \;,\; (f\partial^+) \;,\; (\partial^+ f)$$

The complex Laplacian $\Delta^{\mathbb{C}}$ has the expression

$$\Delta^{\mathbb{C}} = \partial^+\partial = \partial\partial^+ = e_0 \Big(\sum_{\alpha=0}^{m} \frac{\partial^2}{\partial z_\alpha^2} \Big)$$

A function $f : \mathbb{C}_{m+1} \to \mathbb{C}$ holomorphic on $\mathcal{W}$ can be identified with an element $f \in \mathcal{A}(\mathcal{W})$ under the identification $\mathbb{C} \equiv (\mathcal{C}_m^{\mathbb{C}})_0$.

A mapping $f \in \mathcal{A}(\mathcal{W})$ is called left (right) regular if $\partial f = 0$ (resp $f\partial = 0$) , f is called $\mathbb{C}$-harmonic if $\Delta^{\mathbb{C}} f = 0$.

The restriction of $\Delta^{\mathbb{C}}$ on $\mathcal{E}_{m+1}$ (resp. $\mathcal{M}_{m+1}$) gives the negative Laplacián (resp. the wave operator) on $\mathcal{E}_{m+1}$ (resp. $\mathcal{M}_{m+1}$).

Complex harmonic functions correspond to the analytic solutions of the corresponding equations.

Let $N_m = \{ z \in \mathbb{C}_{m+1} \mid \|z\| = 0 \}$ be the complex light cone in $\mathbb{C}_{m+1}$ denote $\mathcal{U} = \mathbb{C}_{m+1} - N_m$. Every element $z \in \mathcal{U}$ is invertible in $\mathcal{C}_m^{\mathbb{C}}$ because

$$z^{-1} = \frac{z^+}{\|z\|}$$

Let us consider the following $\mathcal{C}_m^{\mathbb{C}}$-valued holomorphic forms on $\mathbb{C}_{m+1}$.

$$Dz = \sum_{\alpha=0}^{m} (-1)^\alpha e_\alpha \, d\hat{z}_\alpha \quad , \quad D^+z = e_0 dz_0 + \sum_{j=1}^{m} (-1)^{j+1} e_j dz_j$$

and

$$\Omega = dz_0 \wedge \dots \wedge dz_n \quad \text{,where} \quad d\hat{z}_\alpha = dz_0 \wedge \dots d\hat{z}_\alpha \dots \wedge dz_n \,.$$

Then it is easy to prove the following lemma

L e m m a : If f_1, f_2 are two mappings from $\mathcal{A}(\mathcal{W})$ then

$$\text{(i)}\quad d\,(\,f_1\,Dz\,\,f_2\,) = (\,(f_1\partial)f_2 + f_1\,(\partial f_2)\,)$$
$$\text{(ii)}\quad d\,(\,f_1\,D\,z\,f_2\,) = (\,(f_1\partial^+)f_2 + f_1\,(\partial^+ f_2)\,)$$

If $\mathcal{W} \subset \mathcal{U}$,then mappings

$$G(z) = \frac{z^+}{\|z\|^h} \qquad \text{and} \qquad g(z) = -\,\frac{1}{2(h-1)\,\|z\|^{h-1}}$$

are well defined on $\mathcal{W}$,belong to $\mathcal{A}(\mathcal{W})$ and we have

$$g\,\partial^+ = G \quad , \quad G\partial = 0 \quad \text{on } \mathcal{W} \; .$$

Furthermore if f is C-harmonic function from $\mathcal{A}(\mathcal{W})$,then the (n-1)-form

$$\omega = (\,G\,Dz\,f - g\,D^+z\,\,\partial f\,)$$

is closed on $\mathcal{W}$.

For a point P denote $\mathcal{E}_{m+1}(P)$ the Euclidean subspace of $\mathbb{C}_{m+1}$ shifted to the point P. Let $B_\rho(P)$ be a closed ball with a sufficiently small diameter ρ . Suppose that $B_\rho(P)$ is contained in $\mathcal{W} \cap \mathcal{E}_{m+1}(P)$ and that n-dimensional sphere $S_\rho(P)$ is its boundary.

Now it is possible to present the following Cauchy type integral formula for a solution of the wave equation.

T h e o r e m 1 : Let $\mathcal{W} \subset \mathcal{U}$ be an open set,suppose that $P \in \mathcal{W}$ and let Σ_m be a cycle homological in $\mathcal{W}$ with the sphere $S_\rho(P)$. Then for a complex harmonic function $f \in \mathcal{A}(\mathcal{W})$ we have

$$\text{(C)} \qquad f(P) = \varkappa_m^{-1} \int_{\Sigma_m} \left(\frac{(Q-P)^+}{\|Q-P\|^h}\, DQ\, f(Q) + \frac{1}{2(h-1)\,\|Q-P\|^{h-1}}\, D^+Q\;\; \partial f(Q) \right)$$

where $h = \frac{n+1}{2}$, $\varkappa_m$ is a volume of n-dimensional unit sphere in $\mathcal{E}_{m+1}$.

Proof : The n-form $\omega = (G\,Dz\,f - g\,D^+z\,\partial f\,)$ under the integral sign is closed. From the Stokes theorem it follows that

$$\int_{\Sigma_m} \omega = \int_{S_\rho(P)} \omega \qquad \text{and} \qquad \int_{S_\rho(P)} \omega = \int_{S_\varepsilon(P)} \omega \qquad \text{for} \quad 0 < \varepsilon < \rho \; .$$

Suppose $P = 0$ for simplicity ,then it is possible to prove that

1) if $l(z)$ is a linear approximation of f i.e.

$$l(z) = f(0) + \sum_{\alpha=0}^{m} \frac{\partial f}{\partial z_\alpha}(0)\cdot z_\alpha \quad , \text{ then } \quad \lim_{\varepsilon\to 0^+} \int_{S_\varepsilon(0)} (G\;Dz\;(f-l) -$$

$-g\ D^{+}z\ \ \partial(f-1)\ )\ =0$ and

$$2)\ \lim \int_{S_\varepsilon(0)} (G\ Dz\ 1 - g\ D^{+}z\ \partial 1\ ) = f(0)\cdot \int_{S_\varepsilon(0)} (G\ Dz) =$$
$$= f(0)\ \varkappa_m\ .$$

To add more details to the point 2) ,we can split the integral into two parts :

$$\int_{S_\varepsilon(0)} (G\ DZ\ 1) \qquad \text{and} \qquad \int_{S_\varepsilon(0)} g\ D^{+}z\ \partial 1$$

In the small neighborhood of 0 it is possible take $l(z) = l(0)$ and the second integral can be written in the form

$$\Big(\int_{S_\varepsilon(0)} (\ \ g\ D^{+}z\ \ )\Big)\cdot \partial 1$$

By a direct computation we can see that $\left|\int_{S_\varepsilon(0)} g\ D^{+}z\right| \leq K\varepsilon$ for some K and the integral tends to 0 for $\varepsilon \to 0^{+}$.
So the theorem is proved.

2. THE INTEGRAL FORMULA OF RIESZ .

In [2] M.Riesz presents an integral formula for a solution u of the wave equation in flat Minkowski space.
He showed that the value of u at the point P depends only on the values of u on a surface $\mathcal{S}$,lying inside of negative light cone in P ,and tangential derivatives in points of surface $\mathcal{S}$. Let us recall briefly his formula (for more details see [2]). Suppose again for simplicity that P = 0.

Denote by C the negative light cone with vertex 0,i.e.

$$C = \{\ y \in \mathcal{M}_m \,|\, (y,y) = 0\ ,\ y_0 < 0\ \}$$

Let $\mathcal{S}$ be a (m-1)-dimensional surface in $\mathcal{M}_m$ which is space--like ,then $\mathcal{s} = \mathcal{S} \cap C$ is a (m-2) dimensional surface which is also space-like . If B is a point of $\mathcal{s}$ and b is the position vector of B,then b can be considered as a vector function of some (m-2) parameters $b = b(\lambda_1, \dots ,\lambda_{m-2})$. Clearly $(b,b) = 0$.

Let c be a vector function of the same parameters $\lambda_1 ,\dots, \lambda_{m-2}$ satisfying

(i) $(c,c) = 0$ i.e. c is a null vector

(ii) $(c,db) = 0$ i.e. c is normal to the tangent plane to $\mathcal{s}$ in B.

(iii) $(c,b) = \frac{1}{2}$ i.e. c is normalized with respect to b .

In other words c lies on the generator of the second characteristic surface Ψ (the first one is C) in B going through $\mathscr{S}$.

Further we have

(iv) $(b,db) = (c,dc) = (b,dc) = 0$.

The surface $\mathscr{S}$ is parametrized by

$$y = b(\lambda) + \tau\, c(\lambda)$$

and $y = \sigma\,(b(\lambda) + \tau\, c(\lambda))$ is a parametrization of a certain m-dimensional domain $\mathscr{D}$ in $\mathcal{M}_m$. The metric tensor in the new coordinate system $(\lambda_1, \ldots, \lambda_{m-2}, \tau, \sigma)$ has the form

$$d\,y^2 = \tau\, d\sigma^2 + \sigma\, d\sigma\, d\tau + \sum_{i,k=1}^{m-2} \gamma_{ik}\, d\lambda_i\, d\lambda_k$$

where $\gamma_{ik} = \bar{\gamma}_{ik} \cdot \sigma^2$ and $\bar{\gamma}_{ik}$ does not depend on σ .

Denote

$$\Gamma = ((\gamma_{ij})) \quad , \quad \bar{\Gamma} = ((\bar{\gamma}_{ij}))$$

then the matrix of the metric tensor has the form

$$G = ((\, g_{rs}\,)) = \left(\begin{array}{cc|c} \tau & \frac{1}{2}\sigma & 0 \\ \frac{1}{2}\sigma & 0 & \\ \hline & 0 & \Gamma \end{array}\right)$$

Further denote $\gamma = \det \Gamma$, $\bar{\gamma} = \det \bar{\Gamma}$ and $g = \det G$.

The main role in the description of a solution of the wave equation plays the function

$$F(\tau, \lambda) = \frac{\sqrt{\bar{\gamma}(\tau,\lambda)}}{\sqrt{\bar{\gamma}(0,\lambda)}}$$

It is possible to prove that the following equation holds

$$F(\tau, \lambda) = \frac{\bar{D}(\tau,\lambda)}{\bar{D}(0,\lambda)}$$

where

$$\bar{D}(\tau, \lambda) = \det \begin{pmatrix} b_0, & c_0, & \frac{\partial b_0}{\partial \lambda_1} + \tau \frac{\partial c_0}{\partial \lambda_1}, & \ldots & \frac{\partial b_0}{\partial \lambda_{m-2}} + \tau \frac{\partial c_0}{\partial \lambda_{m-2}} \\ \vdots & \vdots & & & \\ b_{m-1}, & c_{m-1}, & \frac{\partial b_{m-1}}{\partial \lambda_1} + \tau \frac{\partial c_{m-1}}{\partial \lambda_1}, & \ldots, & \frac{\partial b_{m-1}}{\partial \lambda_{m-2}} + \tau \frac{\partial c_{m-1}}{\partial \lambda_{m-2}} \end{pmatrix}$$

Now we can write the integral formula of M.Riesz :

Let u be a solution of the wave equation in a neighborhood of 0 ,then

$$\text{(R)} \quad u(0) = (-\pi)^{1-h} \int_{\mathscr{S}} \frac{\partial^{h-2}}{\partial \tau^{h-2}} \left(\frac{1}{2} \frac{\partial F}{\partial \tau} \cdot u + F \cdot \frac{\partial u}{\partial \tau} \right)\Big|_{\tau=0} ds$$

The value of u at the point 0 thus depends only on the value of u on $\mathscr{S}$ and derivatives of u in the characteristic directions .

3. A COMPARISON OF THE TWO FORMULAS (R) AND (C) .

Let us keep the notation of §1 and §2 . We shall show how the formula (R) can be derived from the formula (C) .

Suppose P = 0 for the simplicity and suppose that u is C-harmonic in a "good" neighborhood $\mathcal{W}$ of $0 \in \mathcal{C}_{m+1}$,which contains a neighborhood of $\mathcal{N}$. Let us consider new coordinates in

$$w_o = z_o \ , \quad w_j = -i\, z_j \ , j = 1,\dots,n \ .$$

In these coordinates we have $\| w \| = (w_o)^2 - \sum_{j=1}^{n} (w_j)^2$,and

$$\mathcal{E}_{m+1} = \left\{ w = \sum_{\alpha=0}^{n} w_\alpha e_\alpha \ \middle|\ \overline{w}_o = -w_o \ , \ \overline{w}_j = w_j , j = 1,\dots n \right\}$$

$$\mathcal{M}_{m+1} = \left\{ w = \sum_{\alpha=0}^{n} w_\alpha e_\alpha \ \middle|\ \overline{w}_\alpha = w_\alpha \ , \alpha = 1,\dots,n \right\}$$

Let $\mathcal{S}_\varepsilon \subset \mathcal{W}$, $\mathcal{S}_\varepsilon \doteq \{ w \mid w = b + \tau c \ , \ \tau \in \mathbb{C} \ , \ \|\tau\| =$
The method described in [1] can be easily generalized to the dimension (n+1) and we get for a sufficiently small ρ , ε that

$S_\rho(0)$ is homological to $\mathcal{S}_\varepsilon$ in $\mathcal{W}$.

Another proof of this fact will be published in [4]

Starting from the formula (C) for the function u ,we have for $\Sigma_m = S_\rho(0)$, ρ sufficiently small

$$u(0) = \varkappa_m^{-1} \int_{S_\rho(0)} \left(\frac{z^+}{\|z\|^h} Dz\, u(z) + \frac{1}{2(h-1)\|z\|^{h-1}} D_z^+ \partial u(z) \right) =$$

$$= \frac{i^{2h-1}}{\varkappa_m} \int_{S_\rho(0)} \left(\frac{\sum_{\alpha=0}^{n} (-1)^\alpha d\hat{w}_\alpha}{\|w\|^h} - \frac{1}{2(h-1)\|w\|^{h-1}} \left(\frac{\partial u}{\partial w_0} d\hat{w}_0 + \sum_{j=1}^{2h-1} \frac{\partial u}{\partial w_j} d\hat{w}_j \right) \cdot e_0 \right.$$

from the Stokes theorem ,after some computations

$$u(0) = \frac{i^{2h-1}}{\varkappa_m} \int_{\mathcal{S}} \left(\int_{|\tau|=\varepsilon} \left(\frac{u}{\tau^h} \bar{D}(\tau,\lambda) + \frac{\partial u}{\partial \tau} \frac{1}{(h-1)\tau^{h-1}} \bar{D}(\tau,\lambda) \right) d\tau \right) d\lambda$$

using the residua formulae and substitution for $\varkappa_m$

$$u(0) = -\, (-\pi)^{1-h} \int_{\mathcal{S}} \frac{\partial^{h-2}}{\partial \tau^{h-2}} \left(\frac{1}{2} u \frac{\partial F}{\partial \tau} + \frac{\partial u}{\partial \tau} F \right) \Big|_{\tau=0} ds$$

which is the formula (R) (for the complex function of real variable) ,the sign -1 follows from the negativity of the norm.

4. INTEGRAL FORMULA FOR LEFT REGULAR MAPPING .

Using the method of §3 we can get the following result .

T h e o r e m 2 : Let $\mathcal{W} \subset \mathcal{U}$ be a "good" neighborhood of $0 \in \mathbb{C}_{m+1}$ and let $\Phi \in \mathcal{A}(\mathcal{W})$ be a left regular mapping. The the following holds

$$\text{(G)} \quad \Phi(0) = -(-\pi)^{1-k} \int_{s} \frac{\partial^{k-1}}{\partial \tau^{k-1}} \left(F \cdot (b^{+} c \cdot \Phi) \right)_{|\tau=0} \, ds$$

where $b = b_0 e_0 + i \sum_{j=1}^{m} b_j e_j$, $c = c_0 e_0 + i \sum_{j=1}^{m} c_j e_j$ are expressed in the Clifford algebra form .

R e m a r k : The formula (G) can be used for the expression of the value of spinor functions which are left regular. This is a generalization of the integral formula for a solution of massless field equation for the spin 1/2 . The proof of the formula (G) and a further investigation and results concerning left regular functions are presented in the forthcoming paper [5] .

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