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A BRIEF INTRODUCTION TO THE MONOPOLE

E. Corrigan

1. Introduction

Many people here are mathematicians, not physicists and may be wondering why anyone should be interested in magnetic monopoles. After all everyone learns at school that there are no such things in nature — all natural magnets are dipoles. Maxwell's equations when we first encounter them reflect this apparent fact,

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{1.1}$$

$$\partial_\mu {}^*F^{\mu\nu} = 0 \tag{1.2}$$

$${}^*F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\ , \quad F_{0i} = E_i\ , \quad F_{ij} = \epsilon_{ijk}B_k \tag{1.2b}$$

where j^ν is an electric current density, and it is used to introduce a convenient vector potential A_μ. Defining

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.3}$$

we see that eq.(1.2) becomes an identity. However, if the electric current is zero we may note that equation (1.1) and (1.2) have a symmetry. The electric field $\underline{E}$ may be replaced by $\underline{E}'$ and the magnetic field $\underline{B}$ by $\underline{B}'$ where

$$\underline{E} = \cos\theta\ \underline{E}' + \sin\theta\ \underline{B}'$$

$$\underline{B} = -\sin\theta\ \underline{E}' + \cos\theta\ \underline{B}' \tag{1.4}$$

for any real θ . We could thus take the point of view that this electric-magnetic symmetry is fundamental for the vacuum and then have to explain why it is broken for the sources, since there is no corresponding current on the right hand side of eq.(1.2).

Much of theoretical physics is involved with postulating symmetries and then understanding the mechanisms that destroy them. The question of electric-magnetic symmetry and the origin of its breaking has stimulated a vast literature on the topic of magnetic monopoles beginning late in the nineteenth century and continuing to the present day(3,7,11).

Historically, the subject breaks naturally into two pieces, pre and post 1974. Before 1974 monopoles were optional. There was no overwhelming reason to predict their existence and the interest in them was because of the problems of their accommodation into existing frameworks — a non-trivial task, as will be briefly explained below. After 1974 events take a different turn. 't Hooft and Polyakov(11) independently pointed out that quantum field theories constructed to unify the strong, weak and electromagnetic interactions actually predict the existence of magnetic monopoles in a surprising way. Indeed, the predicted monopoles are somewhat of an embarrassment. With a mass of around 10^{15} times the mass of a proton, the ability to provoke proton decay(2,24) and their dangerously high abundance in the early moments of the universe(7,8), mechanisms now have to be invented to explain why they are not seen.

2. Pre 1974

Obviously it is not possible in a couple of talks to be comprehensive and cover all the aspects of monopoles. Nevertheless we can get some idea of the difficulties and surprises by looking at the simplest possible case.

Consider the static magnetic field due to a single magnetic monopole of strength g, situated at the origin

$$\underline{B}_M = g\,\underline{x} / 4\pi |\underline{x}|^3 \tag{2.1}$$

Since there are good reasons for supposing that vector potential mentioned above is the fundamental entity for describing the electromagnetic field (both classically and quantum mechanically) we should like to maintain the definition (1.3) as far as possible. However, we clearly cannot write $\underline{B}_M = \nabla \wedge \underline{A}$ for the monopole field, at least not everywhere. Dirac's solution to this problem in 1931(+) was to change the definition of $\underline{B}_M$ to compensate for the necessary use of singular potentials $\underline{A}$ to describe the monopole field. Nowadays, following Wu and Yang (1975)[28] we can paraphrase Dirac's argument in the following way, to reveal a structure well-known to geometers.

Surround the monopole at $\underline{x} = 0$ by a spherical shell of radius $|\underline{x}|$ and divide it into two hemispheres N and S (for North and South) joined by an equator E . On each of the hemispheres let the vector potential be $\underline{A}_N$ and $\underline{A}_S$ respectively, where

$$\underline{A}_N = \frac{g}{4\pi |\underline{x}|} \frac{(1-\cos\theta)}{\sin\theta} \hat{\underline{\phi}} \qquad 0 \le \theta \le \pi/2 \tag{2.2}$$

and

(+) Dirac's 1931 paper is reprinted in reference (7).

$$\underline{A}_S = -\frac{g}{4\pi|\underline{x}|}\frac{(1+\cos\theta)}{\sin\theta}\hat{\underline{\phi}} \qquad \pi/2 \le \theta \le \pi \tag{2.3}$$

(Here θ and ϕ are the usual polar and azimuthal angles on the sphere with axis perpendicular to the plane of E). Clearly, $\underline{A}_N$ is singular at the south pole, $\underline{A}_S$ at the north pole of the sphere. We note also that the curl of $\underline{A}_N$ in the northern hemisphere is $\underline{B}_M$, as is the curl of $\underline{A}_S$ in the south. Hence we can have a non-singular description of the monopole field provided we are prepared to use two patches on the sphere.

On the equator E the potentials $\underline{A}_N$ and $\underline{A}_S$ differ by a 'gauge transformation' i.e.

$$\underline{A}_N - \underline{A}_S = \underline{\nabla}\sigma \tag{2.4}$$

for a suitable choice of σ . The gauge function cannot be continuous around E . If we compute the magnetic flux out of the sphere of radius $|\underline{x}|$ we find

$$g = \int_N d\underline{S}.\underline{B}_M + \int_S d\underline{S}.\underline{B}_M = \oint_E d\underline{x}.(\underline{A}_N - \underline{A}_S) = \sigma(2\pi) - \sigma(0) \tag{2.5}$$

and the discontinuity in σ measures the strength of the pole.

As a hint of the effect of the monopole on quantum mechanics we can consider the wave function for a charged particle moving in the electromagnetic field of the monopole. With the same set up as above the wave function is given on each patch N and S and related in E by the gauge function σ as follows

$$\psi_N = \exp\{iq\sigma/\hbar\}\,\psi_S \tag{2.6}$$

Here, q is the charge of the charged particle, and $\hbar$ is Planck's constant. For the consistency of quantum mechanics the wave function must be continuous. In other words, although σ is necessarily discontinuous in order that the monopole charge can be non zero, $e^{iq\sigma/\hbar}$ must be continuous. We deduce that

$$\frac{qg}{4\pi} = \frac{1}{2} n \hbar , \qquad n \in \mathbb{Z}, \tag{2.7}$$

Dirac's celebrated quantisation condition(2,6). The quantisation condition implies that the existence of a monopole in the universe demands the quantisation of charge (in units of $2\pi\hbar/g$) to maintain the consistency of the quantum theory. To summarise, we can say that although the monopole is optional, to incorporate it demands a better understanding of the electromagnetic formalism and leads to new insight. It is perhaps worth remarking that in 1931 there was no other explanation of charge quantisation.

Much work has been done to construct a complete quantum theory of charges and monopoles both 'first' and 'second' quantised and there is still controversy over the consistency of the latter(7). Rather than dwell on this let us move up to and beyond the developments of 1974.

3. Post 1974

To begin this section we shall first provide a brief sketch of the unification idea and how it works in a simple example. The framework for this lies in the Yang-Mills gauge theory — a generalisation of eqs.(1.1 - 1.3) to the case when the vector potential is not just a set of four real functions, as it is for electromagnetic theory, but rather takes its values in (a representation of) the Lie algebra of a group G (15). Thus

$$A_\mu = \sum_{a=1}^{N} A_\mu^a T^a \quad , \qquad N = \dim G \tag{3.1}$$

where the matrices T^a are a representation of the Lie algebra of G , (normally in what follows the fundamental representation). The generalisation of the field strength tensor $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu]_- \tag{3.2}$$

where e is a constant. Changes of gauge are effected by

$$\begin{aligned} A_\mu &= g^{-1} A'_\mu g + \tfrac{1}{e} g^{-1} \partial_\mu g \\ F_{\mu\nu} &= g^{-1} F'_{\mu\nu} g \quad , \qquad\qquad g(x) \in G \end{aligned} \tag{3.3}$$

If G is the group $U(1)$ eqs.(3.2) and (3.3) clearly reduce to the electromagnetic case, otherwise the nonlinear piece in $F_{\mu\nu}$ is inevitable. The gauge invariant action

$$S = -\int d^4x \ \tfrac{1}{2} \operatorname{tr} \left(F_{\mu\nu} F^{\mu\nu}\right)$$

is stationary when

$$D_\mu F^{\mu\nu} = 0 \tag{3.4}$$

where $D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + e[A_\mu, F^{\mu\nu}]$ is the covariant divergence of the field strength. The equation

$$D_\mu {}^*F^{\mu\nu} = 0 \tag{3.5}$$

is easily checked to be an identity. Eqs.(3.4) and (3.5) provide the generalisations of equations (1.1) and (1.2) to non-abelian gauge groups with no sources.

The unification idea of Glashow, Salam and Weinberg[25] is to treat all the forces between quarks, leptons etc. as being described by the various components of A^a_μ in the same way as the electromagnetic force is described by a single vector potential[16]. However, to achieve it we have to recognise two things. First of all, weak, electromagnetic and strong forces <u>are</u> different at low energy so, which of the components of A^a_μ is to be regarded as the electromagnetic one with all that that entails? Secondly, the photon which is the quantum of the electromagnetic field is massless — as indicated by its long range — and this is clearly not the case for the particles or quanta mediating all the other interactions. At least some of them should be massive. For example, the particles mediating the weak interactions should be about eighty times more massive than the proton. The particles responsible for proton decay have a mass many orders of magnitude greater than this (about 10^{14} x mass of proton). The problem is the above action enjoys too much symmetry and some of it must be removed.

An elegant and useful way to reduce the symmetry is to arrange that although the action of the theory and its field equations are invariant under the gauge group G, nevertheless the vacuum or lowest energy state of the theory is not. In that case, the symmetry is said to break spontaneously down to the subgroup H of G which is the invariance group of the vacuum. The Higgs' mechanism achieves precisely this[24].

Consider the simplest possible example due to Georgi and Glashow : $G = SU(2)$ and $H = U(1)$ [11]. In addition to the three vector potentials A^a_μ we introduce a triplet of scalars ϕ^a to provide source terms in a special way. The energy density of the augmented theory may be written

$$\mathcal{E} = \frac{1}{2}\left[B_i^b B_i^b + E_i^b E_i^b \right] + \frac{1}{2}\left(D_\mu \underline{\phi}\right)^b \left(D_\mu \underline{\phi}\right)^b + \frac{\lambda}{4}\left(\phi^b \phi^b - a^2\right)^2 \tag{3.6}$$

(repeated indices are summed). The lowest energy 'state' ($\mathcal{E} = 0$) satisfies

$$B_i^b = E_i^b = 0 = \left(D_\mu \underline{\phi}\right)^b \quad , \quad \phi^b = a n^b \; , \quad b = 1,2,3 \tag{3.7}$$

where $\underline{n}(x_\mu)$ is a gauge transformation from a constant unit vector. The energy of this field configuration is zero. Since $\underline{n} \neq 0$ it is only invariant under those transformations leaving $\underline{n}$ invariant, a U(1) subgroup of SU(2). Clearly, complete invariance for the 'vacuum' has been lost because of the special choice of potential for the scalar field $\underline{\phi}$. If we look at small fluctuations away from the zero energy solution we see that the combination $\underline{n}.\underline{A}_\mu$ remains massless (there is no $(\underline{n}.\underline{A}_\mu)(\underline{n}.\underline{A}_\mu)$ term in the energy) but the pieces of $\underline{A}_\mu$ perpendicular to $\underline{n}$ obtain a mass $ea\hbar$. They are also charged ($q = \pm e\hbar$) since they couple suitably to the massless component which we interpret as the electromagnetic potential. Effectively, a single component of $\underline{\phi}$ remains with a mass $\sqrt{\lambda}\, a\hbar$. It is uncharged and does not couple to $\underline{n}.\underline{A}_\mu$. To summarise, the spontaneous symmetry breaking changes the spectrum of the theory from a symmetrical set of three massless vector particles to a single massless photon, two charged vector bosons of mass $ea\hbar$ and a single uncharged scalar of mass $\sqrt{\lambda}\, a\hbar$. Undoubtedly, this model is much too simple for nature but it does illustrate the essential points of the trick.

For a more physical theory we would need a larger gauge group (e.g. SU(5)), a more elaborate Higgs mechanism (at least two sets of Higgs' scalars) and to introduce the quark and lepton fields which we have so far ignored. These complications are not really necessary to the understanding of the basic ingredients of the monopole story at the moment and we shall not dwell on them. They are, however, crucial to other aspects of the monopole mentioned in section 1. The variety of monopoles and the ways in which equations such as (2.7) generalise is fascinating.[11,21]

Thus far there is no hint of the magnetic monopoles. Indeed, we even appear to have an explanation of charge quantisation. This is because in the toy model the possible electric charges are multiples of $\frac{1}{2}eh$ as is easily seen by adding other fields in the fundamental representation of SU(2) and seeing how they couple to the photon when the symmetry has broken spontaneously.

However, the monopoles <u>are</u> there, but in disguise. Reconsider the energy density eq.(3.6). The total energy of a field configuration is not necessarily finite but, if it is then asymptotically (as $|\underline{x}| \to \infty$) the fields must approach a vacuum configuration (3.7). Now, suppose $\underline{n}(x)$ is not constant then to arrange the covariant derivative of $\underline{\phi}$ to vanish sufficiently rapidly as $|\underline{x}| \to \infty$ we shall need to have

$$A_\mu^b \sim \frac{1}{ea^2}\left(\partial_\mu \underline{\phi} \wedge \underline{\phi}\right)^b$$

(apart from an unimportant piece proportional to $\underline{\phi}$ which we have discarded) and hence

$$B_k^b \sim \left(-\frac{1}{2ea^3}\,\epsilon_{ijk}\;\underline{\phi}\cdot\partial_i\underline{\phi}\wedge\partial_j\underline{\phi}\right)\phi^b \tag{3.9}$$

We have already remarked that $\underline{n}.\underline{B}_k$ is the Maxwell magnetic field in the spontaneously broken theory and so we see that the asymptotic behaviour of the Higgs' field governs the asymptotic magnetic field. Note that if

$$\phi^b = \pm a \hat{x}^b \tag{3.10}$$

(where $\hat{\underline{x}}$ is the radial unit vector) then the magnetic field is precisely that of a magnetic monopole of charge $\mp 4\pi/e$ (in our units). It is interesting to note that the flux of the magnetic field $\underline{n}.\underline{B}_k$ given by (3.9) is precisely the degree of the mapping $\underline{n}(\underline{x})$, regarded as a map from S_2 (sphere at infinity) to S_2 (set of all unit vectors), and hence measures the homotopy type of $\underline{n}$ (11) .

't Hooft and Polyakov also argued that there should exist time independent solutions to the appropriate field equations having the boundary conditions (3.9) and (3.10) and which are everywhere regular. These solutions have a finite total energy E given by

$$E = \int d^3\underline{x}\ \mathcal{E} = \frac{4\pi a}{e} F(\lambda) \tag{3.11}$$

with $F(o) = 1$. The energy of the field configuration is given predominantly by the ratio of the mass of the heavy vector particles $(ae\hbar)$ divided by the fine structure constant $\hbar e^2/4\pi$ (since empirically $F(\lambda)$ depends weakly on λ , at least for $0 \leq \lambda \leq 10$). Thus the mass of a monopole is typically very large compared with the mass of a proton. (In the SU(5) theory mentioned above the crucial mass is that of the very heavy particle responsible for the conjectured proton decay). The energy density is concentrated near a zero of the Higgs field (at least in the toy model) which can be regarded as defining the position of the monopole. Finally, we note that the magnetic charge of the monopole and the basic electric charge of the theory ($\frac{1}{2}eh$) obey Dirac's quantisation condition.

The fact that the monopoles reappear as solutions to field equations and are not 'put in by hand' as fundamental fields raises many questions which have not been completely answered. From a mathematical point of view it is a very exciting phenomenon because it is the first example of its kind in four dimensions and there are many features similar to behaviour found in two dimensional theories, such as the Sine Gordon equation, exhibiting solitons(15). It is not yet known whether the monopole really is a soliton in the strict sense of the term. It is known that there are many other solutions for the special case $\lambda = 0$ and it is of interest to construct them. Their mathematical structure is intriguing and we shall return to that in the next section, where we shall report briefly on what has been done.

Physically, the monopoles are on a different footing to the other particles of the theory since they are not obviously the quanta of any fundamental field. They are particle-like solutions to classical equations, whereas the other particles only appear when the field theory is quantised. It may well be that this apparent imbalance is a hint of a more elaborate structure to quantum field theory — as is indeed the case for the Sine Gordon theory in two dimensions(4,17) — which must be understood in order to see the 'real' symmetry of the unified theories. Certainly, there are many curious effects associated with them(3,14).

Particularly tantalising is the idea that the monopoles are an indication of a new sort of 'electric'-'magnetic' duality, generalising the idea of section 1, which will only be fully visible at the quantum level(19). In the alternative view of the world the monopoles will be fundamental gauge fields (perhaps) and the electrically charged particles, fundamental from our current way of looking at things, will be the 'solitons'[22].

Although we have used a toy model to infer the existence of monopoles much work has been done to classify their properties within any other gauge group. The interested reader is encouraged to pursue the literature[11,10,21].

4. Exact solutions

If $\lambda = 0$ Prasad and Sommerfield discovered a solution to the Yang-Mills-Higgs equations (with the boundary conditions explained in the last section) in terms of elementary functions. Putting, for convenience, $a = e = h = 1$, their solution is

$$\phi^b = \frac{x^b}{|\underline{x}|}\left(\frac{1}{|\underline{x}|} - \coth|\underline{x}|\right)$$

$$A_0^b = 0$$

$$A_i^b = \epsilon_{bij}\frac{x_j}{|\underline{x}|^2}\left(1 - \frac{|\underline{x}|}{\sinh|\underline{x}|}\right) \tag{4.1}$$

This is a monopole situated at the origin with a total energy of 4π in the new units. It is the tip of the iceberg as far as exact solutions are concerned.

It is possible to prove the existence[14] of further solutions of magnetic charge N units which are static and depend on a total of $4N-1$ gauge invariant parameters[28], ($3N$ of which you can think of, roughly speaking, as the positions of the N monopoles). The energy of these solutions is just $4\pi N$. It is possible to find several different ways to generate these solutions, though none of the ways is totally explicit. It is also possible to prove the existence of other solutions, also static, which may be in some sense bound states of monopoles and antimonopoles[26]. These have an energy definitely greater than zero but less than 8π . No progress has so far been made in understanding them.

If we restrict ourselves to $\lambda = 0$ and seek time-independent fields $\underline{\phi}$, $\underline{A}_i$ (in the gauge choice $\underline{A}_0 = 0$) then we can rewrite the total energy as

$$E = \int d^3\underline{x} \ \tfrac{1}{2} \left(D_i \underline{\phi} - \underline{B}_i \right)^b \left(D_i \underline{\phi} - \underline{B}_i \right)^b \ + 4\pi N \tag{4.2}$$

provided $\underline{\phi}$ satisfies an asymptotic boundary condition appropriate to a monopole solution of total magnetic charge N . Thus, for a given integer N the energy is bounded below by $4\pi N$ and attains the bound if

$$B_i^b = (D_i \underline{\phi})^b \tag{4.3}$$

the Bogomolny equation. In other words, the multiply charged monopole solutions satisfy a first order equation — which indeed implies the relevant second order equations that we have not written down. (The other solutionsmentioned above do not satisfy (4.3) but must satisfy the second order static field equations in a suitable gauge).

Introducing a dummy Euclidean variable x_4 , on which none of the fields A_i^a or ϕ^a depend, we can write $\phi^a = A_4^a$ and $F_{i4}^a = (D_i \phi)^a$. This enables us to reinterpret equation (4.3) as the equation of self duality

$$F^a_{\alpha\beta} = \tfrac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^a_{\gamma\delta} \quad , \quad \alpha,\beta = 1,2,3 \text{ or } 4 \tag{4.4}$$

Eq. (4.4) is already of interest because in four Euclidean dimensions it determines the minima of the SU(2) action functional

$$S = \int d^4x \;\; \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

known as 'instantons'. The action at the local minima is $8\pi^2|k|$, k integer or zero. The two situations, instantons in $\mathbb{R}^4$ and monopoles in $\mathbb{R}^3$ are very different globally, because of the boundary conditions, but locally the field equation (4.4) looks the same. One might expect therefore that methods devised to solve the instanton problem(1,5,13) might be adaptable also to the monopole problem. This is indeed the case, but some ingenuity is required to provide the necessary alterations. At present there are three approaches, each with its merits: the Hungarian approach(9) based on a Riemann-Hilbert formulation of inverse scattering theory, the twistor approach initiated by Ward(1,27,23,6,12) and, finally, the most direct approach from many points of view, the Atiyah, Drinfeld, Hitchin, Manin and Nahm construction (ADHMN)(1,5,20). Here we shall briefly discuss the latter. Details of the others can be found elsewhere.

5. The ADHM construction for instantons

As a warming up exercise we can offer a simple description of the ADHM construction of the instantons for an SU(2) gauge theory. The procedure is easily adapted to larger gauge groups, but the ideas are amply demonstrated via the simplest case. Consider the $k+1$ column vector V ($k \in \mathbb{Z}^+$) whose entries V_i are each 2×2 matrices of the form

$$V_i = v_i^{(0)} + i \sum_{a=1}^{3} v_i^{(a)} \sigma^a , \qquad \left(v_i^{(\mu)}\right)^* = v_i^{(\mu)} \tag{5.1}$$

where the σ 's are the usual Pauli matrices for SU(2). The entries of the vector V are not arbitrary. V is to be chosen orthogonal to a set of k vectors arranged as a k+1 x k matrix Δ , all of whose entries are also 2 x 2 matrices, as (5.1). Thus,

$$\Delta^{+} V = 0 \tag{5.2}$$

Further, Δ is linear in the spatial variables x^{μ} , also coded as a 2 x 2 matrix, i.e.

$$x = x^{4} + i\,\underline{\sigma}.\underline{x} \ , \qquad \Delta = a + b\,x \tag{5.3}$$

Finally, the (anti hermitean) SU(2) gauge potential A_{μ} is defined by

$$A_{\mu} = V^{+} \partial_{\mu} V \tag{5.4}$$

provided V is also normalised to satisfy

$$V^{+} V = 1 . \tag{5.5}$$

It is straightforward to check that the field strength computed from (5.2), (5.3), (5.4) and (5.5) is self dual if and only if the quantity $\Delta^{+}\Delta$ is invertible everywhere, and is the tensor product of a hermitean k x k matrix with the two dimensional unit matrix $\mathbb{1}_{2}$. The latter condition imposes a set of quadratic constraints on the entries of the matrices a and b . The nature of the solutions to these constraints has not been clarified apart from a couple of simple cases. They are more complicated than appears at first sight. It is however possible to count the degrees of freedom of each solution once it is realised that Δ has some redundancy, indeed

$$\Delta' = Q \Delta R \ , \qquad Q \in Sp(k+1) \ , \ R \in GL(k, \mathbb{R}) \tag{5.6}$$

with $\mathbb{Q}$ and $\mathbb{R}$ independent of x^μ will yield the same vector potential. Taking this equivalence and the quadratic constraints into account leaves a total of 8k-3 effective gauge invariant parameters. For k = 1 or 2 the nature of the parameter space is known, not otherwise.

Finally, one ought also to check the value of the action. This can easily be done via a remarkable formula

$$-\tfrac{1}{2}\,\mathrm{tr}\, F_{\mu\nu} F_{\mu\nu} = \tfrac{1}{2}\,\partial^2\partial^2 \ln \det (\Delta^+\Delta)^{-1}, \quad \partial^2 = \partial_\mu\partial_\mu \tag{5.7}$$

which enables the action integral to be computed knowing only the asymptotic behaviour of $(\Delta^+\Delta)^{-1}$ namely

$$(\Delta^+\Delta)^{-1} \sim (b^+b)^{-1}/|x|^2 \tag{5.8}$$

yielding $8\pi^2 k$ as desired. That all instantons can be obtained this way was first proved by ADHM.

6. The ADHMN construction for monopoles (20,12)

In a remarkable sequence of papers Nahm has proposed an extension of the ADHM construction that works for monopoles. For several years it was known only how to construct the BPS monopole (eq.4.1) but, more recently, the methods for dealing with the others have been developed.

In essence, the basic trick is to replace the matrix Δ by a (ordinary) differential operator acting on functions V defined on the interval $[-\frac{1}{2},\frac{1}{2}]$ in a new variable Z . The range of Z lies between the two eigenvalues of the asymptotic Higgs' field, ϕ or A_4 . The inner products, defining for example the vector potential, are redefined as integrals over Z . The potentials are to be independent of x_4 as noted before, but V will depend on x_4 via the factor $e^{ix_4 z}$. Thus, we shall replace (5.5) and (5.4) by

$$\int_{-k_2}^{k_2} dz\, v^+ v = 1$$

$$A_4 = \int_{-k_2}^{k_2} dz\; z\, v^+ v \tag{6.1}$$

$$A_i = \int_{-k_2}^{k_2} dz\; v^+ \partial_i v ,$$

respectively. Ignoring the factor $e^{ix_4 z}$ and making use of a Z dependent transformation Q (eq.5.6) we replace eq.(5.2) by

$$\left(\frac{d}{dz} + \left(\underline{T} + \underline{x} \right) \cdot \underline{\sigma} \right) v = 0 \tag{6.2}$$

where the three T's are each k x k matrices and functions of Z . The latter is inevitable since we need to arrange that the combination $\Delta^+ \Delta$, now a second order differential operator, be invertible and proportional to a k x k matrix quantity times the unit 2 x 2 matrix, as before. A direct calculation shows this to be so provided

$$\underline{T}^+ = \underline{T} \tag{6.3}$$

and

$$\frac{d\underline{T}}{dz} = i\, \underline{T} \wedge \underline{T} , \tag{6.4}$$

another deceptively simple-looking set of equations. Remarkably, eqs.(4.6) are also a set of self duality equations, this time for a set of four 'potentials' T_μ independent of three variables but dependent on the fourth (computed in the gauge $T_4 = 0$). They are a sort of complementary set to the equations of the original monopole problem. We can also recognise eq.(6.2) as a Dirac equation, provided we ignore the $\underline{x}$ piece. Using the same formula as before (5.3) suitably interpreted yields the energy of the monopole to be $4\pi k$. In other words, in a mysterious way the charge of the monopole, hitherto considered to be a topological quantity, is also the order of this complementary gauge group!

There is a danger on inspecting equation (6.2) that there could be more than two independent solutions for V, indeed as many as $2k$, and we do not know which two to take to construct the SU(2) potentials via eqns(6.1). There is also the necessity of arranging the correct number of parameters — $4k-1$. Happily these two requirements are closely related in the following way. The equations for $\underline{T}$, eq(6.4), imply that $\underline{T}$ will usually develop a singularity in Z for some Z and, unless it is a simple pole it will be an essential singularity. The requirement that $\underline{T}$ be no more singular than a simple pole at each end of the interval $-\frac{1}{2} \leq Z \leq \frac{1}{2}$ reduces the effective number of parameters to precisely $4k-1$. Since $\underline{T}$ is singular at $Z = \pm\frac{1}{2}$ then some of the solutions for V will also be singular, and indeed not normalisable, destroying the first of equations (6.1). The others will be acceptable. From a study of eq(6.4) in the vicinity of a simple pole it is clear that the residue of such a pole is a k dimensional representation of the SU(2) Lie algebra. If at each pole the residue is also an irreducible representation the number of solutions to the V equation is precisely two, as required. Thus the boundary conditions in Z are responsible for the choice of complementary gauge group — in this case SU(2).

The solutions to the Dirac equation, in the original four dimensional space in the background of the monopole potential and Higg's field, will have a special dependence on x_4 since all the other fields are independent of this coordinate. Indeed, we can say that the dependence on x_4 must be contained in a factor $e^{ix_4 z}$ and the Dirac equation reads

$$(i\,\underline{\sigma}.\underline{\nabla} + iz + A_4 + i\,\underline{\sigma}.\underline{A})\,\psi = 0. \tag{6.5}$$

Compare this with (6.2). The Dirac equation has k normalisable solutions if the monopole charge is k and Nahm has pointed out the relationship between $\psi(\underline{x}, z)$ and $\underline{T}(z)$. It is summarised by:

$$\underline{T} = \int d^3\underline{x}\ \ \underline{x}\ \psi^+\psi \qquad \text{if} \qquad \int d^3\underline{x}\ \psi^+\psi = 1, \tag{6.6}$$

and bears a striking resemblance to (6.1). It appears that the two Dirac fields V, ψ are also playing complementary roles in defining the corresponding gauge fields A_μ, $\underline{T}$ respectively. (It is difficult not to be impressed by the idea that the expectation value of the coordinate $\underline{x}$ is in some sense a gauge field.) The extent to which this complementarity pervades the ADHMN construction, and whether it can be extended to give some understanding of the situation in which there are two variables upon which nothing depends, is an objective for further study.

So far we have discussed the structure for SU(2) alone. Nahm has also shown how to modify the construction for all other gauge groups as well but the details of that would take us beyond the scope of this introductory talk.

Hopefully, these talks, scant though they have to be compared with the wealth of material on the subject of monopoles, give an idea of the interest behind their study. Many difficult problems remain but to an optimist the achievements so far suggest that more will become known before long, and perhaps the complete story of the mathematics and physics of monopoles will be told in the next few years.

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Instead of a comprehensive list of papers we list a selection of the many reviews and some of the more recent original articles, further references may be found therein.

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