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HOMOTOPY AND HOMOLOGY IN PRETOPOLOGICAL SPACES

Davide Carlo Demaria - Rosanna Garbaccio Bogin

With a process which is similar to the one of the classical case, we can develop a homotopy theory and a singular homology theory for pretopological spaces. This will be used in the continuation of this paper to obtain shape groups and Čech homology groups in another way. In other words, instead to approximate a topological space S by means of polyhedra, we reduce the more the set of admissible functions into S , in such a way to obtain the set of continuous maps.

In fact, among pretopological spaces, the ones given by means of principal filters (i.e. pf-spaces) are particularly interesting for us. The reason is that any topological space (S, T) can be obtained as the inverse limit of a suitable directed family $S = \{(S_i, P_i)\} (i \in J)$ of pf-spaces such that $S_i = S$ for each $i \in J$. (We put $S_i > S_j$ iff the identity from S_i to S_j is precontinuous, etc.). Then, given a point $x \in S$, for any pf-space $S_i \in S$, we can calculate the prehomotopy groups $\Pi_n(S, x)$, and we will put $\check{\Pi}_n(S, x) = \varprojlim \Pi_n(S_i, x)$. Similarly we obtain Čech homology groups.

We observe in advance that, if (S, T) is a compact topological space, the shape groups $\check{\Pi}_n(S, x)$ and Čech homology groups are the classical ones. We only remark that it will be possible to consider finite open coverings and consequently to take pf-spaces belonging to the same homotopy type of finite graphs.

Here we expound a homotopy and homology theory for pretopological spaces, and we examine the properties of pf-spaces.

Finally we observe that finite graphs are pf-spaces and moreover a function from a topological space to a finite graph is precontinuous iff it is regular.

1. PRETOPOLOGICAL SPACES.

1.1 *Definition*⁽¹⁾ Given a nonempty set S , for each point $x \in S$ take a filter F_x in S such that $\overline{F_x} \leq F_x$. The family $P = \{F_x\} (x \in S)$ is called a pretopology in S , and the couple (S, P) is called a pretopological space. We also say " S carries

⁽¹⁾ See [6], page 1374. These spaces are called closure spaces in [1], pseudo-topological spaces in [5].

pretopology P'' , and we simply say " S is a pretopological space" when it is not necessary to specify P explicitly.

If each filter $F_x \in P$ is principal, we say that S is a principally filtered pretopological space or a pf-space.

Remark. Any topological space S may be considered as a pretopological space, putting for each $x \in S$ $F_x = \mathcal{U}_x$, where \mathcal{U}_x is the neighbourhood filter of x .

1.2 Definition Let (S, P) be a pretopological space. For any subset X of S , we put:

$$\begin{aligned} \text{cl}(X) &= \{x \in S / F_x \wedge \overline{X} \neq \emptyset\}; \\ \text{int}(X) &= \{x \in S / F_x \leq \overline{X}\}. \end{aligned}$$

Remark. cl and int are respectively a closure operation and an interior operation such that $\text{int}(X) = S - \text{cl}(S - X)$ for any $X \subseteq S$. (S, P) is a topological space iff $\text{cl}(\text{cl}(X)) = \text{cl}(X)$ for any $X \subseteq S$.

1.3 Definition Let (S, P) and (S', P') be pretopological spaces, and $f: S \rightarrow S'$ a function. f is said precontinuous at $x \in S$ iff $f(F_x) \leq F'_{f(x)}$, where $f(F_x)$ denotes the filter in S' with base $\{f(F)\} (F \in F_x)$. Then we say that $f: S \rightarrow S'$ is a precontinuous map iff f is precontinuous at each $x \in S$.

Remark. Let us consider two subsets X of (S, P) and X' of (S', P') , and two points $x \in X$ and $x' \in X'$. By $f: (S, X) \rightarrow (S', X')$ we will denote a precontinuous map $f: S \rightarrow S'$ such that $f(X) \subseteq X'$. By $f: (S, X, x) \rightarrow (S', X', x')$ we will denote a precontinuous map $f: S \rightarrow S'$ such that $f(X) \subseteq X'$ and $f(x) = x'$.

1.4 Proposition Let (S, P) and (S', P') be pretopological spaces. $f: S \rightarrow S'$ is a precontinuous map iff:

$$(\forall A' \subseteq S') (\forall B' \subseteq S') (A' \cap \text{cl}(B') = \emptyset \Rightarrow f^{-1}(A') \cap \text{cl}(f^{-1}(B')) = \emptyset).$$

Proof: The necessity follows from $\text{cl}(f^{-1}(B')) \subseteq f^{-1}(\text{cl}(B'))$.

Sufficiency: Given $x \in S$ and $F' \in F'_{f(x)}$, we have $\{f(x)\} \cap \text{cl}(S' - F') = \emptyset$. Then $x \notin \text{cl}(f^{-1}(S' - F'))$, and so there is an $F \in F_x$ such that $F \cap f^{-1}(S' - F') = \emptyset$, i.e. $f(F) \subseteq F'$.

1.5 Definition Let S carry pretopology $P = \{F_x\} (x \in S)$, and $X \subseteq S$. Then let P^* be the family $\{F_x^*\} (x \in X)$, where $F_x^* = \{F \cap X\} (F \in F_x)$. We say that (X, P^*) is a subspace of the pretopological space (S, P) .

1.6 Definition Let S' carry pretopology $P' = \{F'_x\} (x \in S')$, and S'' carry pretopology $P'' = \{F''_y\} (y \in S'')$. We call product of (S', P') and (S'', P'') the couple (S, P) , where $S = S' \times S''$, $P = \{F_{(x,y)}\} ((x,y) \in S)$, and $F_{(x,y)}$ is the product filter of F'_x and F''_y .

2. HOMOTOPY OF PRECONTINUOUS MAPS.

As in the classical case, we can develop a homotopy theory.

2.1 Definition Let S and S' be pretopological spaces, and I the unit interval $\{t / 0 \leq t \leq 1\}$ with the standard topology. Two precontinuous maps f and g from S to S' are called homotopic (written $f \sim g$) iff there exists a precontinuous map $H: S \times I \rightarrow S'$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each $x \in S$. Such a function H is called

a prehomotopy of f to g , and we write $H: f \sim g$ to mean " H establishes a homotopy of f to g ".

2.2 Definition Let S and S' be pretopological spaces. We say that S and S' belong to the same homotopy type, or more simply that S and S' are homotopic (written $S \sim S'$), iff we find two precontinuous maps $f: S \rightarrow S'$ and $g: S' \rightarrow S$ such that $gf \sim 1_S$ and $fg \sim 1_{S'}$.

2.3 Definition Let us consider a nonempty set S , a pretopological space (S', P') and a function $f: S \rightarrow S'$. For any $x \in S$, let us denote by F_x^* the filter of base $\{f^{-1}(F')\} (F' \in \mathcal{F}_{f(x)}')$. The family $P_x^* = \{F_x^*\} (x \in S)$ is called the pretopology induced in S from (S', P') by f^{-1} .

2.4 Proposition If the function f from the set S to the pretopological space (S', P') is surjective, then (S, P^*) and (S', P') belong to the same homotopy type.

Proof: Clearly $f: (S, P^*) \rightarrow (S', P')$ is precontinuous. We obtain a precontinuous map $g: (S', P') \rightarrow (S, P^*)$, choosing for any $x' \in S'$ a point $g(x') \in f^{-1}(x')$. Clearly $fg = 1_{S'}$. Moreover, since $F_x^* = \mathcal{F}_{gf(x)}^*$ for any $x \in S$, we define a prehomotopy H of gf to 1_S putting:

$$H(x, t) = \begin{cases} x & \text{for } t=1, x \in S; \\ gf(x) & \text{for } t < 1, x \in S. \end{cases}$$

3. PREHOMOTOPY GROUPS.

Let S be a pretopological space and $a \in S$. For each positive integer n , we consider the set $C_n(S, a)$ of all precontinuous n -loops based at a , i.e. the set of all precontinuous maps $f: I^n \rightarrow S$ such that $f(I^n) = \{a\}$, where I^n is the unit n -cube with the standard topology and \dot{I}^n is the boundary of I^n .

Then we define the homotopy of precontinuous n -loops based at a , and we give a group structure to the quotient set $\Pi_n(S, a)$. We call $\Pi_n(S, a)$ the n -dimensional prehomotopy group of S based at a .

The group $\Pi_n(S, a)$ is abelian for $n \geq 2$. Moreover, if b is another point of S and there exists a precontinuous path starting at a and ending at b , then the groups $\Pi_n(S, a)$ and $\Pi_n(S, b)$ are isomorphic.

Then, given $X \subseteq S$ and $a \in X$, we define the set $C_n(S, X, a)$ of relative precontinuous n -loops of the triple (S, X, a) and we construct the n -dimensional relative prehomotopy group $\Pi_n(S, X, a)$.

Afterwards we obtain a homomorphism $\partial: \Pi_n(S, X, a) \rightarrow \Pi_{n-1}(X, a)$.

Then, given two triples (S, X, a) and (T, Y, b) of pretopological spaces, we define an operation \star , which associates to any precontinuous map $\phi: (S, X, a) \rightarrow (T, Y, b)$ a homomorphism $\phi_\star: \Pi_n(S, X, a) \rightarrow \Pi_n(T, Y, b)$ for each dimension n .

So we obtain the prehomotopy system $\{\Pi, \partial, \star\}$. The following six conditions are satisfied:

(I) If $\phi: (X, A, a) \rightarrow (X, A, a)$ is the identity, then $\phi_\star: \Pi_n(X, A, a) \rightarrow \Pi_n(X, A, a)$ is the

identical isomorphism.

(II) Let $\phi: (X, A, a) \rightarrow (Y, B, b)$ and $\psi: (Y, B, b) \rightarrow (Z, C, c)$ be precontinuous maps. Then

$$(\psi\phi)_* = \psi_* \phi_*.$$

(III) Let $\phi: (X, A, a) \rightarrow (Y, B, b)$ be a precontinuous map and let $\psi: (A, a) \rightarrow (B, b)$ be the restriction of ϕ to A . Then the following diagram commutes:

$$\begin{array}{ccc} \Pi_n(X, A, a) & \xrightarrow{\partial} & \Pi_{n-1}(A, a) \\ \phi_* \downarrow & & \downarrow \psi_* \\ \Pi_n(Y, B, b) & \xrightarrow{\partial} & \Pi_{n-1}(B, b) \end{array}$$

(IV) Let us consider a triple (X, A, a) of pretopological spaces, the homomorphism $i_*: \Pi_n(A, a) \rightarrow \Pi_n(X, a)$ induced by the canonical injection $i: (A, a) \rightarrow (X, a)$, and the homomorphism $j_*: \Pi_n(X, a) \rightarrow \Pi_n(X, A, a)$ induced by the natural injection j from $C_n(X, a)$ to $C_n(X, A, a)$. The following sequence is exact:

$$\dots \xrightarrow{\partial} \Pi_n(A, a) \xrightarrow{i_*} \Pi_n(X, a) \xrightarrow{j_*} \Pi_n(X, A, a) \xrightarrow{\partial} \Pi_{n-1}(A, a) \xrightarrow{i_*} \dots \xrightarrow{\partial} \Pi_1(A, a) \xrightarrow{i_*} \Pi_1(X, a)$$

(V) If the precontinuous maps $\phi: (X, a) \rightarrow (Y, b)$ and $\psi: (X, a) \rightarrow (Y, b)$ are homotopic, then $\phi_* = \psi_*$.

(VI) If $X = \{a\}$, all groups $\Pi_n(X)$ are null.

The proofs are similar to the ones given in the classical case (see for example [4]). Generally the fibration axiom is not valid; but this is not important for our next considerations.

4. HOMOLOGY GROUPS.

As in the classical case, we can develop a singular homology theory.

4.1 *Definition* Let us consider a pretopological space S and the standard euclidean p -simplex Δ_p . We call singular p -simplex (or more simply p -simplex) on S any precontinuous map from Δ_p to S .

We obtain the p -dimensional homology group $H_p(S)$ of the pretopological space S , and then, for any subset X of S , the p -dimensional relative homology group $H_p(S, X)$ of the pair (S, X) . The boundary homomorphism $\partial: C_p(S) \rightarrow C_{p-1}(S)$ induces a homomorphism $\partial: H_p(S, X) \rightarrow H_{p-1}(X)$.

Afterwards, given two pretopological spaces S' and S'' , $X' \subseteq S'$ and $X'' \subseteq S''$, we define an operation $*$, which associates to any precontinuous map $\phi: (S', X') \rightarrow (S'', X'')$ a homomorphism $\phi_*: H_p(S', X') \rightarrow H_p(S'', X'')$ for each dimension p .

So we obtain the homology system $\{H, \partial, *\}$. The following seven conditions hold:

(1) If $\phi: (X, A) \rightarrow (X, A)$ is the identity, then $\phi_*: H_p(X, A) \rightarrow H_p(X, A)$ is the identical isomorphism.

(2) Let $\phi: (X, A) \rightarrow (Y, B)$ and $\psi: (Y, B) \rightarrow (Z, C)$ be precontinuous maps. Then $(\psi\phi)_* = \psi_* \phi_*$.

(3) Let $\phi: (X, A) \rightarrow (Y, B)$ be a precontinuous map and $\psi: A \rightarrow B$ the restriction of ϕ to A . Then the following diagram commutes:

$$\begin{array}{ccc} H_p(X, A) & \xrightarrow{\phi_*} & H_p(Y, B) \\ \partial \downarrow & & \downarrow \partial \\ H_{p-1}(A) & \xrightarrow{\psi_*} & H_{p-1}(B) \end{array}$$

(4) Given a pair (X, A) , let us consider the homomorphism $i_*: H_p(A) \rightarrow H_p(X)$ induced by the canonical injection $i: A \rightarrow X$, and the homomorphism $j_*: H_p(X) \rightarrow H_p(X, A)$ induced by the natural injection $j: Z_p(X) \rightarrow Z_p(X, A)$. The following sequence is exact:

$$\dots \xrightarrow{\partial} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X, A) \xrightarrow{\partial} H_{p-1}(A) \xrightarrow{i_*} \dots \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A)$$

(5) If the precontinuous maps $\phi: (X, A) \rightarrow (Y, B)$ and $\psi: (X, A) \rightarrow (Y, B)$ are homotopic, then $\phi_* = \psi_*$.

(6) If $X = \{a\}$, then all homology groups $H_p(X)$ are null.

(7) Let A and U be nonempty subsets of a pretopological space X , such that $\text{cl}(U) \subseteq \text{int}(A)$. Then the canonical injection $i: (X - U, A - U) \rightarrow (X, A)$ induces an isomorphism $i_*: H_p(X - U, A - U) \rightarrow H_p(X, A)$.

The proofs of Conditions (1)-(6) are similar to the corresponding ones given for the classical singular homology (see for example [7]).

Proof of (7): For any p -simplex σ on the pretopological space X , $\{\sigma^{-1}(X - U), \sigma^{-1}(A)\}$ is an open covering of Δ_p , since $\{X - \text{cl}(U), \text{int}(A)\}$ is a covering of X , $X - \text{cl}(U) = \text{int}(X - U)$ and moreover $\sigma^{-1}(\text{int}(Y)) \subseteq \sigma^{-1}(\overline{Y})$ for any $Y \subseteq X$.

Now let us denote by a_0, a_1, \dots, a_p the vertices of Δ_p and consider a baricentric subdivision $B^r(a_0, a_1, \dots, a_p) = \Sigma \alpha_i \tau_i$ of the identity $(a_0, a_1, \dots, a_p): \Delta_p \rightarrow \Delta_p$, with $r > 0$ such that $\alpha_i \neq 0$ implies $\text{diam} \tau_i < \epsilon$, where ϵ is a positive real number such that either $V(y, \epsilon) \subseteq \sigma^{-1}(X - U)$ or $V(y, \epsilon) \subseteq \sigma^{-1}(A)$ for any $y \in \Delta_p$. Then, if $\alpha_i \neq 0$, the simplex $\sigma_i = \sigma \tau_i$ on X is such that either $\sigma_i(\Delta_p) \subseteq X - U$ or $\sigma_i(\Delta_p) \subseteq A$. Therefore, each element of $H_p(X, A)$ is represented by a relative p -cycle α which is a linear combination of p -simplices $\sigma \in C_p(X)$ such that either $\sigma(\Delta_p) \subseteq X - U$ or $\sigma(\Delta_p) \subseteq A$. So we can show that the homomorphism i_* is surjective and that $\text{ker } i_* = 0$ (see [7], excision theorem).

Remark. Sometimes, for relative homotopy and relative homology, we will have to consider a subset X of a pretopological space (S, P) with a pretopology P_X finer than the one induced in X by P . We can repeat all foregoing considerations except excision theorem.

5. PF-SPACES.

Let S carry pretopology $P = \{\overline{A_x}\} (x \in S)$, where $\overline{A_x}$ denotes the principal filter of base A_x . From (S, P) we obtain a directed graph G whose vertex set is S . To define G we say that there is the directed arc xy iff $x \neq y$ and $y \in A_x$. (Instead, saying that there is the directed arc xy iff $x \neq y$ and $x \in A_y$) we obtain the dually directed graph G^* . G is called the graph of the pretopological space (S, P) .

Viceversa, let G be a directed graph with vertex set S . For any vertex x , we consider the following sets:

$$A(x) = \{\{x\} \cup \{y \in S \mid \exists xy\};$$

$$A^*(x) = \{\{x\} \cup \{y \in S \mid \exists yx\}.$$

$P_G = \{\overline{A(x)}\} (x \in S)$ and $P_{G^*} = \{\overline{A^*(x)}\} (x \in S)$ are pretopologies on S .

Remark that, if G is the graph of a pf-space (S, P) , then $\overline{A(x)} = F_x$ for any

$x \in S$. So the pretopological spaces (S, P_G) and (S, P) are isomorphic.

5.1 *Definition* A covering \mathcal{R} of a pretopological space (S, P) is an interior covering iff for each $x \in S$ there is an $X \in \mathcal{R}$ such that $x \in F_x$ (see [1]).

5.2 *Definition* Let S be a pretopological space. S is compact iff any interior covering of S has a finite subcovering (see [1]). S is locally compact iff for each $x \in S$ we find a compact subspace X of S such that $x \in F_x$.

5.3 *Definition* A function f from a pretopological space S to a directed graph G is o -regular (o^* -regular) iff the following conditions hold:

- (1) if v, w are distinct vertices of G and there is not the directed arc vw , then $f^{-1}(v) \cap cl(f^{-1}(w)) = \emptyset$ (resp. $cl\{f^{-1}(v)\} \cap f^{-1}(w) = \emptyset$);
- (2) if X is a compact subset of S , then $f(X)$ is finite.

5.4 *Proposition* Let us consider a pretopological space S , a directed graph G , and a function $f: S \rightarrow G$. We have the following three statements:

- (a) Let the graph G be locally finite. If f is a precontinuous map from S to (G, P_G) (resp. to (G, P_G^*)), then f is o -regular (resp. o^* -regular).
- (b) Let the pretopological space S be locally compact. If $f: S \rightarrow G$ is o -regular (resp. o^* -regular), then f is a precontinuous map from S to (G, P_G) (resp. to (G, P_G^*)).
- (c) Let the graph G be finite. Then f is a precontinuous map from S to (G, P_G) (resp. to (G, P_G^*)) iff f is o -regular (resp. o^* -regular).

Proof: (a). Let G be locally finite and $f: S \rightarrow (G, P_G)$ precontinuous.

(1). If $v \neq w$ and there is not the arc vw , we have $w \notin A(v)$. But $w \notin A(v) \Leftrightarrow v \notin cl(\{w\}) \Leftrightarrow f^{-1}(v) \cap cl(f^{-1}(w)) = \emptyset$.

(2). For any vertex v of G , let us put $X_v = f^{-1}(A(v))$, and observe that $f^{-1}(v) \subseteq \subseteq f^{-1}(\text{int}(A(v))) \subseteq \text{int}(X_v)$. Hence, for any $X \subseteq S$, the family $\{X_v\}_{v \in f(X)}$ is an interior covering of X . If X is compact, $\{X_v\}_{v \in f(X)}$ has a finite subcovering $\{X_{v_1}, X_{v_2}, \dots, X_{v_n}\}$; therefore $f(X) \subseteq \bigcup_{i=1}^n A(v_i)$. Hence $f(X)$ is finite, because $A(v)$ is finite for any $v \in G$.

(b). Let S be locally compact and $f: S \rightarrow G$ be o -regular.

For any subset Z' of (G, P_G) , the family $\{f^{-1}(w)\}_{w \in Z'}$ is locally finite. In fact, for any $x \in S$ there is an $X \in F_x$ which is compact, and $X \cap f^{-1}(w) \neq \emptyset$ only for a finite set of vertices $w \in G$, since $f(X)$ is finite. Therefore (see [1]) $cl(f^{-1}(Z')) = \bigcup_{w \in Z'} cl(f^{-1}(w))$. Now, if $Y' \subseteq G$ is such that $Y' \cap cl(Z') = \emptyset$, we have $f^{-1}(v) \cap cl(f^{-1}(w)) = \emptyset$ for any $v \in Y'$ and $w \in Z'$. So $f^{-1}(v) \cap cl(f^{-1}(Z')) = \emptyset$, and hence $f^{-1}(Y') \cap cl(f^{-1}(Z')) = \emptyset$.

(c). Let the graph G be finite. "Only if" part follows from (a). "If": observe that the family $\{f^{-1}(w)\}_{w \in Z'}$ is finite for any $Z' \subseteq G$.

Remark. Now suppose that the pf-space (S, P) is such that, for any $x, y \in S$, $x \in A_y \Leftrightarrow y \in A_x$. In this case the pf-space is called symmetrical, and its graph is undirected.

Viceversa, let G be an undirected graph. We obtain a symmetrical pf-space (G, P_G)

putting $P_G = \{\overline{viox}\} (x \in G)$, where $viox$ is the set containing x and all vertices y of G that are consecutive to x .

A function f from a pretopological space S to the graph G is called regular iff the following two conditions hold:

- (1) $f^{-1}(v) \cap \text{cl}(f^{-1}(w)) = \text{cl}(f^{-1}(v)) \cap f^{-1}(w) = \emptyset$ for any pair (v, w) of non consecutive distinct vertices of G ;
- (2) if $X \subset S$ is compact, then $f(X)$ is finite.

We can prove that, if the undirected graph G is locally finite, the prehomotopy groups $\Pi_n((G, P_G), v)$ are the regular homotopy groups $Q_n(G, v)$ (see [3]).

6. EXAMPLES.

Given a covering $\mathcal{R} = \{A_i\} (i \in J)$ of a nonempty set S , we will define two pretopologies in S . To this purpose, we consider for each $x \in S$ the sets $A(x, \mathcal{R}) = \bigcap \{A_i \in \mathcal{R} / x \in A_i\}$ and $\text{St}(x, \mathcal{R}) = \bigcup \{A_i \in \mathcal{R} / x \in A_i\}$. Then we denote by (S, R_\cap) and (S, R_\cup) the pf-spaces, whose pretopologies are $R_\cap = \{\overline{A(x, \mathcal{R})}\} (x \in S)$ and $R_\cup = \{\overline{\text{St}(x, \mathcal{R})}\} (x \in S)$.

Now we define a function $\phi: S \rightarrow \mathcal{P}(J)$ putting $\phi(x) = \{i \in J / x \in A_i\}$ for any $x \in S$. Then we consider the set $Y = S / \rho$, where ρ is the equivalence relation in S given by $x \rho x' \Leftrightarrow \phi(x) = \phi(x')$, and the canonical projection $\pi: S \rightarrow Y$. ϕ induces a function $\Phi: Y \rightarrow \mathcal{P}(J)$, which is given by $\Phi(y) = \phi(x)$ where $y \in Y$ and $x \in \pi^{-1}(y)$.

From the pf-space (S, R_\cap) we obtain a directed graph G^\cap , whose vertex set is Y . To define G^\cap , we say that, given $y, y' \in Y$, there is the directed arc yy' iff $\Phi(y)$ is a proper subset of $\Phi(y')$.

Instead, from the simmetrical pf-space (S, R_\cup) we obtain an undirected graph G^\cup , whose vertex set is Y . To define G^\cup , we say that $y, y' \in Y$ are consecutive iff $y \neq y'$ and $\Phi(y) \cap \Phi(y') \neq \emptyset$.

6.1 Proposition The pretopological spaces (S, R_\cap) and (G^\cap, P_{G^\cap}) are homotopic.

Proof: In fact R_\cap is the pretopology induced in S from (G^\cap, P_{G^\cap}) by π^{-1} .

6.2 Proposition The pretopological spaces (S, R_\cup) and (G^\cup, P_{G^\cup}) are homotopic.

Proof: In fact R_\cup is the pretopology induced in S from (G^\cup, P_{G^\cup}) by π^{-1} .

Now assume that the covering $\mathcal{R} = \{A_i\} (i \in J)$ is star-finite, and for any $j \in J$ denote by $\psi(j)$ the finite set $\{i \in J / A_i \cap A_j \neq \emptyset\}$.

6.3 Proposition The graph G^\cap is locally finite.

Proof: Let $y, y' \in Y$.

In G^\cap there is the arc $y'y$ iff $\Phi(y') \subset \Phi(y)$. But $\Phi(y)$ is finite; so the set $\{y' \in Y / \exists y'y\}$ is finite.

Then in G^\cap there is the arc yy' iff $\Phi(y) \subset \Phi(y')$. But $\Phi(y') \subseteq \psi(j)$ for any $j \in \Phi(y)$; so also the set $\{y' \in Y / \exists yy'\}$ is finite.

6.4 Proposition The graph G^\cup is locally finite.

Proof: Let y and y' be distinct vertices of G^\cup . $y' \in viox$ iff $\Phi(y) \cap \Phi(y') \neq \emptyset$; therefore $\Phi(y') \subseteq \psi(j)$ for some $j \in \Phi(y) \cap \Phi(y')$, and hence the set $viox$ is finite.

Now, under the assumption that the covering \mathcal{R} of S is star-finite, we define a partial ordering relation \leq in Y , putting $y \leq y'$ iff $\phi(y) \subseteq \phi(y')$. Then we consider the induced subgraph G' of G^U , whose vertex set is the set Y' of all maximal elements in (Y, \leq) .

6.5 *Proposition* The pretopological spaces (G^U, P_{G^U}) and $(G', P_{G'})$ belong to the same homotopy type.

Proof: For any $y \in Y$ there is some $y' \in Y'$ such that $y \leq y'$. In fact we have $\phi(y') \subseteq \psi(j)$ for any $j \in \phi(y)$. So we define a precontinuous map $f: G^U \rightarrow G'$, choosing for each $y \in Y$ an $y' \in Y'$ such that $y \leq y'$. Then we consider the canonical injection $g: G' \rightarrow G^U$. Clearly $fg = 1_{G'}$. Moreover we obtain a prehomotopy H of 1_{G^U} to gf , putting:

$$H(y, t) = \begin{cases} gf(y) & \text{for } t=1, y \in Y; \\ y & \text{for } t < 1, y \in Y. \end{cases}$$

Remark. If the covering \mathcal{R} of S is star-finite, the pf-spaces (S, R_U) and $(G', P_{G'})$ are homotopic by Propositions 6.4 and 6.5.

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