## Heinz-Dietrich Doebner; Jiří Tolar Quantization on submanifolds

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#### QUANTIZATION ON SUBMANIFOLDS

H. D. Doebner and J. Tolar

ABSTRACT. Extrinsic position and momentum operators are defined on a Riemannian manifold (M,g) via an isometric embedding i: (M,g)-- $(\mathbb{R}^n, g_n)$ . An intrinsic Hamiltonian  $\widetilde{H}_0$  describing free dynamics on (M,g) is constructed by an associated Riemannian submersion satisfying a physically well justified condition.

#### 1. MOTIVATION

We consider a physical system moving on a smooth manifold. Such a situation arises e.g. in the classical description of a many-particle system in terms of collective coordinates which are associated with its motion in the large. Then one deals in fact with an embedding i of a manifold M of configurations described by the collective coordinates, in a configuration manifold -  $R^n$  in most cases - of the system as a whole. In quantum mechanics, the aim of such a procedure is to isolate a physically distinguished subsystem of a given system modelling more or less accurately the motion of the system when only the collective degrees of freedom are excited.

In non-relativistic classical mechanics, the kinetic energy part of the dynamics in  $\mathbb{R}^n$  is described by the Euclidean metric  $g_n$ ; then the classical kinetic energy on the submanifold M is given by the induced Riemannian metric  $g = i^*g_n$ . In Section 4 we present a solution to the corresponding quantum dynamical embedding problem by using a Riemannian submersion  $\sigma : (W,g_n) \longrightarrow (M,g)$ ,  $W \subset \mathbb{R}^n$ , associated with the isometric embedding  $i:(M,g) \longrightarrow (\mathbb{R}^n,g_n)$  and satisfying a physically well justified requirement II. This results in the free Hamiltonian on (M,g) proportional to the intrinsic Laplace-Beltrami operator, improving our earlier formulations [3], [4].

2. QUANTUM MECHANICS ON  $(\mathbb{R}^n, g_n)$ Let  $(\mathbb{R}^n, g_n)$  be the n-dimensional Euclidean space, i.e. the space  $\mathbb{R}^n$  with global Cartesian coordinates  $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$  for  $\mathbf{r} \in \mathbb{R}^n$ and endowed with the Euclidean Riemannian structure  $g_n$ . Quantum mechanics of a system moving freely in  $(\mathbb{R}^n, g_n)$  is based on the physically well justified canonical position operators  $\mathbb{Q}_j$ , momentum operators  $\mathbb{P}_j$ ,  $j = 1, \ldots, n$ , and the free Hamiltonian  $\mathbb{H}_0$ . They act as e entially self-adjoint operators on a dense invariant domain  $\mathbb{C}_0^\infty(\mathbb{R}^n) \subset \mathbb{L}^2(\mathbb{R}^n, \mathbb{d}^n\mathbf{r})$  as

and

$$(Q_{j}f)(r) = r_{j}f(r), \quad P_{j}f(r) = -i\frac{\partial}{\partial r_{j}}f(r), \quad f \in C_{0}^{\infty}(\mathbb{R}^{n})$$
$$H_{0} = \gamma \sum_{j=1}^{\infty} P_{j}^{2} = -\gamma \Delta_{n};$$

 $\Delta_{\sim}$  is the Laplace-Beltrami operator on  $(\mathbb{R}^n, g_n)$ .

3. THE CONSTRUCTION OF POSITION AND MOMENTUM OPERATORS ON (M,g) BY ISOMETRIC EMBEDDING

Consider an embedding i:  $M \neq q \mapsto iq \epsilon R^n$ , where  $m = \dim M < n$ . It induces a Riemannian metric g on M via

 $g(X,Y) = (i^{\epsilon}g_{n})(X,Y) = g_{n}(i_{\epsilon}X,i_{\epsilon}Y)$ , X, Y  $\epsilon$  TM, where  $i_{\epsilon}: TM \rightarrow TR^{n}$  is the induced mapping of the tangent bundles. Hence (M,g) is a Riemannian manifold and i is an isometric embedding (M,g)  $\rightarrow (R^{n},g_{n})$ .

The embedding i permits to define natural restriction of the Hilbert space  $L^2(\mathbb{R}^n, d^n r)$  via the pull-backs  $\tilde{f}$ ,  $d\mu$  of the wave functions f and of the Lebesgue measure  $d^n r$ , respectively. The resulting Hilbert space is  $L^2(\mathbb{M}, d\mu)$  where  $\mu$  is the Riemann measure associated with g.

We want to define position and momentum operators  $\tilde{Q}_j$  and  $\tilde{P}_j$ on iM acting as symmetric operators on  $C_0^{\mathscr{O}}(iM) \subset L^2(iM, d_{\mathcal{V}})$ . The definition of  $\tilde{Q}_j$ ,  $j = 1, \ldots, n$ , is obvious:

 $(\widetilde{q}_j \widetilde{f})(iq) = (iq)_j \widetilde{f}(iq)$ ,  $\widetilde{f} \in C_0^{\infty}(iM)$ , with  $(iq)_j$  being the Cartesian coordinates of  $iq \in iM$  in  $\mathbb{R}^n$ . Since, in general, there exists no global coordinate system on M, it is not possible to define global position operators on M which act on  $C_0^{\infty}(M)$  as the Q; act on  $C_0^{\infty}(\mathbb{R}^n)$ . Our operators  $\widetilde{Q}_j$  belong to the class of smooth position operators Q(F),  $F \in C^{\infty}(iM)$ , introduced in [9] and defined by

 $[Q(F) \tilde{f}](iq) = F(iq)\tilde{f}(iq), \quad \tilde{f} \in C_{0}^{\infty}(iM).$ 

To define  $\tilde{P}_{j}$  on iM we first observe that the isometric embedding i induces, via  $i_{\star}: TM \rightarrow TR^{n}$ , at each iq  $\epsilon$  iM a decomposition  $T_{iq}R^{n} = T_{iq}iM \oplus N_{iq}$ , where the normal subspace  $N_{iq}$  is specified uniquely as the  $g_n$ -orthogonal complement of  $T_{iq}iM$  in  $T_{iq}R^n$ . Now  $P_j$ 's are proportional to the constant vector fields  $E_j = \partial_j = \partial/\partial r_j$  in  $R^n$ . If  $E_j$ 's are perpendicularly projected at each iq onto  $T_{iq}iM$ , we obtain n vector fields  $\tilde{E}_j = B_j^k \partial_k \in \mathcal{K}(iM)$ , where  $(B_j^k)$  is at each iq a perpendicular projection nxn-matrix of rank m < n. Concerning linear independence of vector fields  $\tilde{E}_j$  we have the following

Lemma 1. The constant vector fields  $E_j$ , j = 1, ..., n, yield by perpendicular projection  $E_j \mapsto \widetilde{E}_j$  exactly m linearly independent vectors at each point iq e iM.

Proof follows from the observation that  $(B_j^k)$  is a projector of constant rank m [8].

R e m a r k 1. In general there are more than m non-vanishing vectors at iq, i.e. sufficiently many, but linearly dependent. Conversely, let a family  $\{X_j\}$  of n vector fields be given on (M,g)which span  $T_qM$  for each q  $\in M$ . We may then ask whether they can be obtained by projection from some orthonormal vector fields on  $(R^n, g_n)$ . The answer is positive [7] if the components of  $X_j$  in a g-orthonormal basis in  $T_qM$  at each q  $\in M$  form an nxm matrix with rows being  $g_n$ -orthonormal. Then n-m further rows can obviously be added to it to form an orthogonal nxn matrix. The resulting vector fields are in general not constant, so the construction is of local nature; it yields a family of orthonormal vector fields which do not vanish only in a tubular neighbourhood of iM in  $R_n$ . A global statement would require special topological tools.

Then  $\widetilde{P}_j$ 's are defined as the unique symmetric operators in  $L^2(\mathbb{M}, d\mu)$  with the 1st order differential operator parts  $-i\widetilde{E}_j$ . We have

Lemma 2.  $\tilde{P}_{j} = -i \left( B_{j}^{k} \partial_{k} + \frac{m}{2} \eta_{j} \right)$  on  $C_{0}^{\infty}(iM)$ , where  $m = \dim M$  and  $\eta_{j}$  are Cartesian components of the mean curvature normal of (M,g) in  $(R^{n},g_{n})$ .

Proof is based on the formula [5], [1]  $\widetilde{P}_{j} = P(\widetilde{E}_{j}) = -i(\widetilde{E}_{j} + \frac{1}{2} \operatorname{div} \widetilde{E}_{j}),$ where the divergence with respect to  $\mu$  can be expressed in terms of the second fundamental tensor of (M,g) in  $(\mathbb{R}^{n},g_{n})$  [8]  $\operatorname{div} \widetilde{E}_{j} = \sqrt[p]{}_{a} \mathbb{B}_{j}^{a} = \mathbb{H}_{a,j}^{a} = m^{?}_{j}.$ 

# 4. THE CONSTRUCTION OF THE HAMILTONIAN ON (M,g) BY RIEMANNIAN SUBMERSION

Considered as an operator in  $L^2(\mathbb{R}^n, \mathbb{d}^n r)$ , the free Hamiltonian  $H_0$  is proportional to the Laplace-Beltrami operator  $\Delta_{\sim}$  on  $(\mathbb{R}^n, g_n)$ .

Then a suitable restriction of  $\Delta_{\mu}$  to the submanifold (M,g) will be identified with  $\widetilde{H}_0$ . However, because the restriction has to be performed via a submersion [2], it will depend strongly on geometrical properties of its fibres. We put two physically motivated requirements to arrive at the unique result:

- I. In order to achieve correct decoupling of the dynamics on iM from  $\mathbb{R}^n$ -iM we assume that submersion  $\sigma$  is Riemannian, i.e.  $\sigma_{\!_{\!\!\mathcal{X}}}$  preserves the lengths of horizontal vectors [6].
- II. In order that the Hamiltonian  $\widetilde{H}_0$  be a candidate for a meaningful quantum observable on M we assume that  $\widetilde{H}_0$  is symmetric in  $C_0^{\infty}(M) \subset L^2(M,\mu)$ .

For a given but otherwise arbitrary submersion  $\sigma$ : W — M, where W is a tubular neighbourhood of iM in R<sup>n</sup> [6], the restriction  $\Delta_{\sigma}$  of  $\Delta_{\sim}$  is defined as

$$(\Delta_{m}f)(r) = (\Delta_{\kappa}\tilde{f})\circ\sigma(r)$$

where  $f = \tilde{f} \cdot \sigma \in C^{\infty}(W)[2]$ , [10]. Since many submersions  $\sigma$  are in general possible,  $\Delta_{\sigma}$  is not unique. This non-uniqueness is present even for a Riemannian submersion and is explicitly stated in

Lemma 3. Let  $\sigma: (W,g_n) \rightarrow (M,g)$ ,  $W \subset \mathbb{R}^n$ , be a Riemannian submersion. Then the restriction  $\Delta_{\sigma}$  of  $\Delta_{\tau}$  is given by

 $\Delta_{\sigma} = \Delta + (n-m)\theta \quad \text{on } C_{o}^{\infty}(M),$ where  $\Delta$  is the Laplace-Beltrami operator on (M,g) and  $\theta$  is the mean curvature vector field of the fibres  $\sigma^{-1}(q)$  at points  $iq \ \epsilon t M$ .

Proof consists in retaining the omitted term in Wallach's proof of his Proposition 7.1 [10].

Adding further requirement II., we arrive at the

The orem. Let M be a paracompact C<sup> $\sigma$ </sup>-manifold. Let ( $\mathbb{R}^n$ ,  $g_n$ ) be the Riemannian Euclidean space and  $\mathcal{A}_n$  its Laplace-Beltrami operator. Consider an embedding i:  $\mathbb{M} \longrightarrow (\mathbb{R}^n, g_n)$  which endows M with a Riemannian structure  $g = i^{*}g_n$ . Let  $\sigma$  be a Riemannian submersion mapping a tubular neighbourhood W of iM onto M. Then the following assertions are equivalent:

(i)  $\Delta_{c}$  is a symmetric operator on  $C_{o}^{\infty}(M) \subset L^{2}(M, \mu)$ ;

(ii) the mean curvature vector field  $\theta$  of the fibres of  $\sigma$  vanishes at the points iq  $\epsilon$  iM;

(iii)  $\Delta_{\sigma} = \Delta$  on  $C_{0}^{\infty}(M)$ .

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Proof becomes trivial after decomposing  $\Delta_{f}$  into the symmetric and skew-symmetric parts

$$\Delta_{\sigma} = (\Delta - \frac{n-m}{2}\operatorname{div} \theta) + (n-m)(\theta + \frac{1}{2}\operatorname{div} \theta)$$

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Assume  $\theta + \frac{1}{2} \operatorname{div} \theta = 0$  on M and  $\theta \neq 0$ ,  $\operatorname{div} \theta \neq 0$  only on a compact subset  $U \subset M$ . Taking in  $C_0^{\infty}(M)$   $\tilde{f} = \operatorname{const.} \neq 0$  in  $U_0 \subset U$ ,  $f \rightarrow 0$  smoothly in U-U<sub>0</sub>, then  $(\theta + \frac{1}{2} \operatorname{div} \theta) \tilde{f} \neq 0$  on U<sub>0</sub> leads to contradiction. Thus (i)  $\Leftrightarrow$  (ii). The other implications are obvious.

R e m a r k 2. Usually only the sufficient condition is used that the fibres  $\sigma^{-1}(q)$  are totally geodesic submanifolds in  $(W,g_n)$ . Being totally geodesic is equivalent to the vanishing of the second fundamental tensor, hence implies  $\theta = 0$ , but not conversely. See [3],[4],[2],[10].

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H.D. DOEBNER

INSTITUT FÜR THEORETISCHE PHYSIK DER TECHNISCHEN UNIVERSITÄT CLAUSTHAL LEIBNIZSTR: 3392 CLAUSTHAL FED. REP. GERMANY

J. TOLAR

DEPARTMENT OF PHYSICS FACULTY OF NUCLEAR SCIENCE AND PHYSICAL ENGINEERING CZECH TECHNICAL UNIVERSITY BŘEHOVÁ 7 115 19 PRAHA 1 CZECHOSLOVAKIA