Heinz-Dietrich Doebner; Jiří Tolar

Quantization on submanifolds

In: Zdeněk Frolík (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 3. pp. [127]--132.

Persistent URL: http://dml.cz/dmlcz/701299

QUANTIZATION ON SUBMANIFOLDS

H. D. Doebner and J. Tolar

ABSTRACT. Extrinsic position and momentum operators are defined on a Riemannian manifold (M,g) via an isometric embedding $i: (M,g) \to (R^n, g_n)$. An intrinsic Hamiltonian $\tilde{H}_0$ describing free dynamics on (M,g) is constructed by an associated Riemannian submersion satisfying a physically well justified condition.

1. MOTIVATION

We consider a physical system moving on a smooth manifold. Such a situation arises e.g. in the classical description of a many-particle system in terms of collective coordinates which are associated with its motion in the large. Then one deals in fact with an embedding i of a manifold M of configurations described by the collective coordinates, in a configuration manifold - R^n in most cases - of the system as a whole. In quantum mechanics, the aim of such a procedure is to isolate a physically distinguished subsystem of a given system modelling more or less accurately the motion of the system when only the collective degrees of freedom are excited.

In non-relativistic classical mechanics, the kinetic energy part of the dynamics in R^n is described by the Euclidean metric g_n; then the classical kinetic energy on the submanifold M is given by the induced Riemannian metric $g = i^* g_n$. In Section 4 we present a solution to the corresponding quantum dynamical embedding problem by using a Riemannian submersion $\sigma : (W, g_n) \to (M,g)$, $W \subset R^n$, associated with the isometric embedding $i: (M,g) \to (R^n, g_n)$ and satisfying a physically well justified requirement II. This results in the free Hamiltonian on (M,g) proportional to the intrinsic Laplace-Beltrami operator, improving our earlier formulations [3], [4].

2. QUANTUM MECHANICS ON (R^n, g_n)

Let (R^n, g_n) be the n-dimensional Euclidean space, i.e. the space R^n with global Cartesian coordinates $(r_1,\dots,r_n)$ for $r \in R^n$ and endowed with the Euclidean Riemannian structure g_n. Quantum mechanics of a system moving freely in (R^n,g_n) is based on the physically well justified canonical position operators Q_j, momentum operators P_j, $j = 1,\dots,n$, and the free Hamiltonian H_o. They act as e entially self-adjoint operators on a dense invariant domain $C_o^\infty(R^n) \subset L^2(R^n, d^n r)$ as

$$(Q_j f)(r) = r_j f(r), \quad P_j f(r) = -i \frac{\partial}{\partial r_j} f(r), \quad f \in C_o^\infty(R^n)$$

and

$$H_o = \gamma \sum_{j=1}^{n} P_j^2 = -\gamma \Delta_n;$$

Δ_n is the Laplace-Beltrami operator on (R^n,g_n).

3. THE CONSTRUCTION OF POSITION AND MOMENTUM OPERATORS ON (M,g) BY ISOMETRIC EMBEDDING

Consider an embedding i: $M \ni q \longmapsto iq \in R^n$, where $m = \dim M < n$. It induces a Riemannian metric g on M via

$$g(X,Y) = (i^* g_n)(X,Y) = g_n(i_* X, i_* Y), \quad X, Y \in TM,$$

where $i_*: TM \to TR^n$ is the induced mapping of the tangent bundles. Hence (M,g) is a Riemannian manifold and i is an isometric embedding $(M,g) \to (R^n,g_n)$.

The embedding i permits to define natural restriction of the Hilbert space $L^2(R^n, d^n r)$ via the pull-backs $\tilde{f}$, $d\mu$ of the wave functions f and of the Lebesgue measure $d^n r$, respectively. The resulting Hilbert space is $L^2(M, d\mu)$ where μ is the Riemann measure associated with g.

We want to define position and momentum operators $\tilde{Q}_j$ and $\tilde{P}_j$ on iM acting as symmetric operators on $C_o^\infty(iM) \subset L^2(iM, d\mu)$. The definition of $\tilde{Q}_j$, $j = 1,\dots,n$, is obvious:

$$(\tilde{Q}_j \tilde{f})(iq) = (iq)_j \tilde{f}(iq), \quad \tilde{f} \in C_o^\infty(iM),$$

with $(iq)_j$ being the Cartesian coordinates of $iq \in iM$ in R^n. Since, in general, there exists no global coordinate system on M, it is not possible to define global position operators on M which act on $C_o^\infty(M)$ as the Q_j act on $C_o^\infty(R^n)$. Our operators $\tilde{Q}_j$ belong to the class of smooth position operators $Q(F)$, $F \in C^\infty(iM)$, introduced in [9] and defined by

$$[Q(F)\tilde{f}](iq) = F(iq)\tilde{f}(iq), \quad \tilde{f} \in C_o^\infty(iM).$$

To define $\tilde{P}_j$ on iM we first observe that the isometric embedding i induces, via $i_*: TM \to TR^n$, at each $iq \in iM$ a decomposition $T_{iq}R^n = T_{iq}iM \oplus N_{iq}$, where the normal subspace N_{iq} is specified uniquely as the g_n-orthogonal complement of $T_{iq}iM$ in $T_{iq}R^n$. Now P_j's are proportional to the constant vector fields $E_j = \partial_j = \partial/\partial r_j$ in R^n. If E_j's are perpendicularly projected at each iq onto $T_{iq}iM$, we obtain n vector fields $\tilde{E}_j = B^k_j \partial_k \in \mathcal{X}(iM)$, where (B^k_j) is at each iq a perpendicular projection nxn-matrix of rank $m < n$. Concerning linear independence of vector fields $\tilde{E}_j$ we have the following

L e m m a 1. The constant vector fields E_j, $j = 1,\dots,n$, yield by perpendicular projection $E_j \mapsto \tilde{E}_j$ exactly m linearly independent vectors at each point $iq \in iM$.

P r o o f follows from the observation that (B^k_j) is a projector of constant rank m [8].

R e m a r k 1. In general there are more than m non-vanishing vectors at iq, i.e. sufficiently many, but linearly dependent. Conversely, let a family $\{X_j\}$ of n vector fields be given on (M,g) which span T_qM for each $q \in M$. We may then ask whether they can be obtained by projection from some orthonormal vector fields on (R^n, g_n). The answer is positive [7] if the components of X_j in a g-orthonormal basis in T_qM at each $q \in M$ form an nxm matrix with rows being g_n-orthonormal. Then n-m further rows can obviously be added to it to form an orthogonal nxn matrix. The resulting vector fields are in general not constant, so the construction is of local nature; it yields a family of orthonormal vector fields which do not vanish only in a tubular neighbourhood of iM in R_n. A global statement would require special topological tools.

Then $\tilde{P}_j$'s are defined as the unique symmetric operators in $L^2(M, d\mu)$ with the 1st order differential operator parts $-i\tilde{E}_j$. We have

L e m m a 2. $\tilde{P}_j = -i\left(B^k_j \partial_k + \frac{m}{2}\eta_j\right)$ on $C^\infty_0(iM)$, where $m = \dim M$ and η_j are Cartesian components of the mean curvature normal of (M,g) in (R^n, g_n).

P r o o f is based on the formula [5],[1]

$$\tilde{P}_j = P(\tilde{E}_j) = -i\left(\tilde{E}_j + \tfrac{1}{2}\operatorname{div}\tilde{E}_j\right),$$

where the divergence with respect to μ can be expressed in terms of the second fundamental tensor of (M,g) in (R^n, g_n) [8]

$$\operatorname{div}\tilde{E}_j = \nabla_a B^a_j = H^{\cdot a}_{a \cdot j} = m\eta_j .$$

4. THE CONSTRUCTION OF THE HAMILTONIAN ON (M,g) BY RIEMANNIAN SUBMERSION

Considered as an operator in $L^2(R^n, d^n r)$, the free Hamiltonian H_o is proportional to the Laplace-Beltrami operator Δ_n on (R^n, g_n).

Then a suitable restriction of Δ_{n} to the submanifold (M,g) will be identified with $\widetilde{H}_0$. However, because the restriction has to be performed via a submersion [2], it will depend strongly on geometrical properties of its fibres. We put two physically motivated requirements to arrive at the unique result:

I. In order to achieve correct decoupling of the dynamics on iM from R^n-iM we assume that submersion σ is Riemannian, i.e. σ_* preserves the lengths of horizontal vectors [6].

II. In order that the Hamiltonian $\widetilde{H}_0$ be a candidate for a meaningful quantum observable on M we assume that $\widetilde{H}_0$ is symmetric in $C_0^{\infty}(M) \subset L^2(M,\mu)$.

For a given but otherwise arbitrary submersion $\sigma : W \to M$, where W is a tubular neighbourhood of iM in R^n [6], the restriction Δ_σ of Δ_n is defined as

$$(\Delta_n f)(r) = (\Delta_\sigma \tilde{f}) \circ \sigma(r)$$

where $f = \tilde{f} \circ \sigma \in C_0^{\infty}(W)$ [2], [10]. Since many submersions σ are in general possible, Δ_σ is not unique. This non-uniqueness is present even for a Riemannian submersion and is explicitly stated in

L e m m a 3. Let $\sigma : (W,g_n) \to (M,g)$, $W \subset R^n$, be a Riemannian submersion. Then the restriction Δ_σ of Δ_n is given by

$$\Delta_\sigma = \Delta + (n-m)\,\theta \qquad \text{on } C_0^{\infty}(M),$$

where Δ is the Laplace-Beltrami operator on (M,g) and θ is the mean curvature vector field of the fibres $\sigma^{-1}(q)$ at points $iq \in iM$.

P r o o f consists in retaining the omitted term in Wallach's proof of his Proposition 7.1 [10].

Adding further requirement II., we arrive at the

T h e o r e m . Let M be a paracompact C^{∞}-manifold. Let (R^n, g_n) be the Riemannian Euclidean space and Δ_n its Laplace-Beltrami operator. Consider an embedding $i: M \to (R^n,g_n)$ which endows M with a Riemannian structure $g = i^* g_n$. Let σ be a Riemannian submersion mapping a tubular neighbourhood W of iM onto M. Then the following assertions are equivalent:

(i) Δ_σ is a symmetric operator on $C_0^{\infty}(M) \subset L^2(M,\mu)$;

(ii) the mean curvature vector field θ of the fibres of σ vanishes at the points $iq \in iM$;

(iii) $\Delta_\sigma = \Delta$ on $C_0^{\infty}(M)$.

P r o o f becomes trivial after decomposing Δ_σ into the symmetric and skew-symmetric parts

$$\Delta_\sigma = \left(\Delta - \frac{n-m}{2}\operatorname{div}\theta\right) + (n-m)\left(\theta + \frac{1}{2}\operatorname{div}\theta\right) .$$

Assume $\theta + \frac{1}{2}\operatorname{div}\theta = 0$ on M and $\theta \neq 0$, div $\theta \neq 0$ only on a compact subset $U \subset M$. Taking in $C_0^{\infty}(M)$ $\tilde{f}$ = const. $\neq 0$ in $U_0 \subset U$, $f \to 0$ smoothly in $U-U_0$, then $(\theta + \frac{1}{2}\operatorname{div}\theta)\tilde{f} \neq 0$ on U_0 leads to contradiction. Thus (i) $\Leftrightarrow$ (ii) . The other implications are obvious.

R e m a r k 2. Usually only the sufficient condition is used that the fibres $\sigma^{-1}(q)$ are totally geodesic submanifolds in (W, g_n). Being totally geodesic is equivalent to the vanishing of the second fundamental tensor, hence implies $\theta = 0$, but not conversely. See [3], [4], [2], [10].

ACKNOWLEDGEMENT

We are grateful to Prof. W. GREUB and Dr. J. HENNIG for numerous stimulating discussions and valuable remarks.

REFERENCES

[1] ANGERMANN B., DOEBNER H.D., TOLAR J. "Quantum kinematics on smooth manifolds", in: Proceedings of the 5th Bulgarian Summer School on Mathematical Problems in Quantum Field Theory, Primorsko 1980, Lecture Notes in Mathematics, Springer-Verlag, Berlin 1982.

[2] BERGER M., GAUDUCHON P., MAZET E. "Le Spectre d'une Variété Riemannienne", Lecture Notes in Mathematics, Vol. 194, Springer-Verlag, Berlin 1971.

[3] DOEBNER H.D., TOLAR J. "Construction of position and momentum operators and Hamiltonians by embedding and submersion", Lett. Math. Phys, 6 (1982), 183-188 and 511 (Addendum).

[4] DOEBNER H.D., TOLAR J. "Quantum mechanics on homogeneous spaces", J. Math. Phys., 16 (1975), 975-984.

[5] DOEBNER H.D., TOLAR J. "On global properties of quantum systems", in: Symmetries in Science, eds. B. Gruber and R.S. Millman, Plenum Press, New York 1980, 475-486.

[6] GROMOLL D., KLINGENBERG W., MEYER W. "Riemannsche Geometrie im Grossen", Lecture Notes in Mathematics, Vol. 55, Springer-Verlag, Berlin 1968.

[7] HENNIG J., private communication.

[8] SCHOUTEN J.A. "Ricci Calculus: An Introduction to Tensor Analysis and Its Geometrical Applications", 2nd edition, Springer-Verlag, Berlin 1954.

[9] SEGAL I.E. "Quantization of nonlinear systems", J. Math. Phys. 1 (1960), 468-488.

[10] WALLACH N.R. p. 1 in "Symmetric Spaces; Short Courses Presented at Washington University", eds. W.M. Boothby and G.I. Weiss, Marcel Dekker, Inc., New York, 1972 (Proposition 7.1).

H.D. DOEBNER
INSTITUT FÜR THEORETISCHE PHYSIK
DER TECHNISCHEN UNIVERSITÄT CLAUSTHAL
LEIBNIZSTR.
3392 CLAUSTHAL
FED. REP. GERMANY

J. TOLAR
DEPARTMENT OF PHYSICS
FACULTY OF NUCLEAR SCIENCE AND PHYSICAL ENGINEERING
CZECH TECHNICAL UNIVERSITY
BŘEHOVÁ 7
115 19 PRAHA 1
CZECHOSLOVAKIA