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CONVERGENT SEQUENCES IN $\beta X$

Roman Frič and Peter Vojtáš

ABSTRACT. Our aim is to construct a completely regular Hausdorff topological space $X$ in which no nontrivial sequence converges and in its Čech-Stone compactification $\beta X$ there is a nontrivial convergent sequence. We show that all three possibilities occur: (IN-OUT) the sequence is in $X$ and its limit point is in $\beta X-X$, (OUT-IN) the sequence is in $\beta X-X$ and its limit point is in $X$ and, finally, (OUT-OUT) both the sequence and its limit point are in $\beta X-X$. We discuss the minimal cardinality of the spaces in question.

Let $X$ be a completely regular Hausdorff space and let $C^*(X)$ be the set of all bounded continuous functions on $X$. Then a sequence $\langle x_n \rangle$ converges in $X$ to a point $x \in X$ iff for each $f \in C^*(X)$ we have $\lim f(x_n) = f(x)$. A sequence $\langle x_n \rangle$ is said to be fundamental whenever $\langle f(x_n) \rangle$ is a convergent sequence for all $f \in C^*(X)$. Clearly, a fundamental sequence $\langle x_n \rangle$ either converges in $X$ or $\bigcup_{n \in \omega} \{x_n\}$ is a closed discrete subset of $X$. If each fundamental sequence converges in $X$, then $X$ is said to be sequentially complete. Realcompact and normal spaces are sequentially complete (cf. [3]).

Proposition 1. If $|X| = \omega$, then there is no convergent sequence in $\beta X$ of the types IN-OUT or OUT-OUT.

Proof. If $|X| = \omega$, then $X$ is normal and hence sequentially complete. Thus no sequence $\langle x_n \rangle$ of points $x_n \in X$ can converge to a point $x \in \beta X-X$. Similarly, if $\langle x_n \rangle$ is a one-to-one sequence of points $x_n \in \beta X-X$, then $Y = X \cup \{x \in \beta X; x = x_n, n \in \omega \}$ is also a sequentially complete space. Thus $\langle x_n \rangle$ cannot converge in $\beta Y = \beta X$ to a point $x \in \beta Y-Y$. Consequently, the sequence $\langle x_n \rangle$ cannot converge in $\beta X$ to a point $x \in \beta X-X$. 


1. IN-OUT

Our construction of a space $X$ in which there is a sequence $(x_n)$ converging in $\beta X$ to a point in $\beta X \setminus X$ and in $X$ no nontrivial sequence converges is based on the following idea.

First, let $\alpha > \omega$ be a cardinal number and let $Y = \omega \times (\alpha + 1)$. Define a topology for $Y$ : all points $[n, \beta]$ for $n \in \omega$ and $\beta \in \alpha$ are isolated; a local base at $[n, \beta]$ for $n \in \omega$ is formed by sets $\{[n, \beta] \cup (K_n - S) : K_n = \{[n, \beta] \in Y; \beta \in \alpha\}$ and $S$ is a countable subset of $K_n$. Then $Y$ is a completely regular Hausdorff space and for each $f \in C(Y)$ we have $f([n, \beta]) = f([n, \beta])$ for all but countably many $\beta \in \alpha$. Note that no nontrivial sequence converges in $Y$.

Second, embed $Y$ into a completely regular Hausdorff space $X$ so that no nontrivial sequence converges in $X$, the sequence $(x_n)$ is a fundamental sequence in $X$, and the set $\{[n, \beta] \in X ; n \in \omega\}$ is a closed discrete subset of $X$. Then $(x_n)$ is an IN-OUT sequence.

At the Winter School we have presented the following space $X$, communicated to us by P. Simon.

**Example 1.** Consider the set $X = ((\omega + 1) \times (2^\omega + 1)) - \{[\omega, 2^\omega]\}$. Define a topology for $X$ :

(i) All points $[n, \beta]$ for $n \in \omega$ and $\beta \in 2^\omega$ are isolated; (ii) For $n \in \omega$ a local base at $[n, 2^\omega]$ is formed by sets $\{[n, \beta] \in X; \beta \in 2^\omega + 1\} - S$, where $S$ is a countable subset of the set $\{[n, \beta] \in X; \beta \in 2^\omega\};$

(iii) Let $h$ be a one-to-one mapping of $2^\omega$ onto $\{U \in G(\omega^\omega) ; |U| = 2\}$ (for $\beta \in 2^\omega$, $h(\beta) = \{F, G\}$, where $F$ and $G$ are distinct uniform ultrafilters on $\omega$). For $\beta \in 2^\omega$, $\{F, G\} = h(\beta)$, $F \in F$ and $G \in G$, the sets $\{[\omega, \beta]\} \cup \{[n, \beta] \in X; n \in F \cup G\}$ form a local base at $[\omega, \beta]$.

It follows from the construction that $X$ is a completely regular Hausdorff space in which no nontrivial sequence converges. Clearly, $Y$ (with $\alpha = 2^\omega$) is a subspace of $X$. Further, $(x_n, 2^\omega)$ is a fundamental sequence in $X$ and $\{[n, 2^\omega] \in X; n \in \omega\}$ is a closed discrete subset of $X$. Consequently, the sequence $(x_n, 2^\omega)$ converges in $\beta X$ to a point in $\beta X \setminus X$.

Here we present another construction of the space $X$ (with no nontrivial convergent sequences) in which $Y$ (with $\alpha = K$) is embedded.

**Example 2.** In [1] it is shown that for
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\[ \kappa = \min \{ \gamma : \text{the Boolean algebra } \mathcal{G}(\omega)/\text{fin is not } (\gamma, \omega, 2) \text{ distributive} \} \]

there is a matrix \{ $P_\alpha : \alpha \in \kappa$ \} such that the following conditions hold:

1. $P_\alpha$ is a maximal almost disjoint family of subsets of $\omega$;
2. $\alpha < \beta$ implies $P_\beta$ refines $P_\alpha$;
3. for each infinite subset $x$ of $\omega$ there is $\alpha \in \kappa$ such that
   \[ |\{ y \in P_\alpha : y \subseteq x \}| = 0 . \]

For each $\alpha \in \kappa$ define
\[ F_\alpha = \{ x \in \omega : |\{ y \in P_\alpha : |y - x| = \kappa \} | < \kappa \} . \]

Clearly, $F_\alpha$ is a filter on $\omega$. Consider the set $X = ((\omega + 1) \times \chi(\kappa + 1)) - \{ [\omega, \kappa] \}$. The topology for $X$ is defined analogously as in Example 1: (i) and (ii) remain and (iii) is replaced by

(iii)* for $\beta \in \kappa$, $F \in F_\beta$, the sets $[\omega, \beta] \cup \{ [n, \beta] : n \in F \}$
form a local base at $[\omega, \beta]$.

Recall that $\omega_1 \leq \kappa \leq \omega < 2^{\omega}$, and so the cardinality of this space is $\kappa < 2^{\omega}$.

At the Winter School we have asked what is the minimal cardinality of the space $X$ in which no nontrivial sequence converges and in $X$ there is an IN-OUT sequence. In [4] it is shown that the minimal cardinality of such a space is $\omega_1$. The construction is of the same type as in the above two examples. In the construction $\alpha = \omega_1$ and $X$ is the set $((\omega + 1) \times (\omega + 1)) - \{ [\omega, \omega_1] \}$ equipped with a topology in which neighborhoods of $[\omega, \beta]$, $\beta \in \omega_1$ are constructed via sums of Fréchet filters.

2. OUT-IN

Example 3. Consider the set $X = (\omega \times \omega) \cup \{ \infty \}$ equipped with the following topology: all points $[n, m] \in \omega \times \omega$ are isolated; a local base at $\infty$ is formed by sets $[\infty] \cup \{ [m, n] \in \omega \times \omega : m > m_0, n > n_0 \}$ - $S$,
where $m_0, n_0 \in \omega$ and $S$ is a subset of $\omega \times \omega$ containing finitely many points in each row and finitely many points in each column of $\omega \times \omega$. Then $X$ is a countable completely regular Hausdorff space in which no nontrivial sequence converges. For each $n \in \omega$ $[n, \omega]$ is homeomorphic to the closure in $\beta X$ of the discrete closed set $K_n = \{ n \} \times \omega$, the homeomorphism being fixed on $\omega$. It is easy to see that if $x_n \in 1_{\beta X} K_n$ - $K_n$, then the sequence $\langle x_n \rangle$ converges in $\beta X$ to the point $\infty$. Since $X$ is countable, it follows from Proposition 1 that there are no (nontrivial) IN-OUT or OUT-OUT sequences in $\beta X$. 
3. OUT-OUT

In our talk at the Winter School we have presented a space (having no nontrivial convergent sequences) for which there are both IN-OUT and OUT-OUT sequences. The space itself has been constructed by tying together a sequence of distinct copies of the space \( X \) from Example 1. We have also announced that we are able to construct a space (having cardinality \( \omega \)) for which there are only OUT-OUT sequences. We present the construction below (Example 4). After the Winter School, during a short visit of W. S. Watson in Košice, we have constructed several spaces (with no nontrivial convergent sequences) having cardinality \( \omega_1 \) for which there are only OUT-OUT sequences. This, together with Proposition 1, shows that \( \omega_1 \) is the minimal cardinality of such spaces. For details see [74].

Example 4. In this construction we use the following observation about \( \omega^\kappa \). It is known ([2]) that each point of \( \omega^\kappa \) is a \( \sigma \)-point (e.g. equivalently, for each nontrivial ultrafilter \( j = \{ x_\alpha ; \alpha \in \kappa \} \) on \( \omega \) there is an almost disjoint refinement (i.e. a system \( \{ y_\alpha ; \alpha \in \kappa \} \) such that \( y_\alpha \subseteq x_\alpha \) and \( \alpha \neq \beta \) implies \( |y_\alpha \cap y_\beta| < \kappa \))

A nontrivial ultrafilter \( j \) on \( \omega \) is said to be a \( \sigma \)-o-point if the following holds: Let \( \{ X_\alpha ; \alpha \in \kappa \} = [j] \omega \) be an enumeration of all countable subsets of \( j \). Then there is an almost disjoint family \( \{ y_\alpha ; \alpha \in \kappa \} \) on \( \omega \) such that for each \( \alpha \in \kappa \) and each \( x \in X_\alpha \) we have \( y_\alpha \subseteq \omega^\kappa \) (modulo finite). Using a slight modification of Hindman's proof (see [5]) of the existence of \( \sigma \)-o-points we can prove the existence of a \( \sigma \)-o-point.

Proposition 2. There are always \( \sigma \)-o-points in \( \omega^\kappa \); assuming CH or MA or RP (Roitman principle), all points of \( \omega^\kappa \) are \( \sigma \)-o-points.

We do not know whether in ZFC each point of \( \omega^\kappa \) is a \( \sigma \)-o-point.

Construction. Let \( j \) be a \( \sigma \)-o-point and let \( X^j, y^j \) be as above. For \( \alpha \in \kappa \), enumerate \( X^j = \{ x^j_\alpha ; n \in \omega \} \) take the product \( R^j = \prod_{\alpha \in \kappa} (x^j_\alpha \cap y^j_\alpha) \). Then \( R^j \) is isomorphic to \( \omega^{\omega} \). As \( \kappa \) (from Example 2) is less or equal to the smallest size of an unbounded family in \( \omega^\omega \), ordered modulo finite (see [1]), there is a strictly increasing sequence of one-to-one functions \( \{ f^j_\alpha , \beta < \kappa \} \subseteq R^j \).

Clearly, for \( j \neq \kappa \) we have \( |f^j_\alpha \cap f^j_\beta| < \kappa \).

Consider the set \( X = \omega \times \omega \cup \omega \). Define a topology for \( X \):

(i) All points \( [n,m] \) for \( n,m \in \omega \) are isolated;

(ii) Let \( h \) be a one-to-one mapping from \( \omega \) onto \( \omega \times \kappa \) and let \( \alpha, \beta, \gamma \) be such that \( h(\gamma) = [\alpha, \beta] \). For \( F \in \mathcal{F}_\beta \) (the very
filter from Example 2) the sets \( \gamma \cup \{ [n, \varepsilon_s^{(n)}(n)]; n \in F \} \) form a local base at the point \( \gamma \).

Then the closure of the set \( \nu_n = \{ [n, m]; m \in \omega \} \) in \( \beta X \) contains \( J_n \), the copy of the \( \varepsilon \)-c-point \( j \). Then \( < J_n > \) is a fundamental sequence and \( \beta X \) is a "pure OUT-OUT" space.

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