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In: Zdeněk Frolík (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 3. pp. [161]–164.

Persistent URL: http://dml.cz/dmlcz/701305

AN INDUCTIVE APPROACH TO THE PIROGOV-SINAI THEORY

Roman Kotecký , David Preiss

The aim of this note is to present an alternative approach to the Pirogov-Sinai theory concerning the description of phase diagrams of lattice systems [1]. Though our motivation is to generalize the Pirogov-Sinai theory so as to include another type of models not covered by the original theory, we use the occasion to present it here in the simplest form, namely exactly in the situation considered by Pirogov and Sinai. This also allows us to keep their notation. Actually we shall use the notation from Sinai's book [2]. The sole exception is that we prefer (mainly in view of future generalization) to consider "nonrelative Hamiltonian" partition functions $Z(\Gamma^q|\beta\mathcal{H})$, $Z_q(V|\beta\mathcal{H})$ related to theirs [2, Def.2.7, 2.8] by

$$Z(\Gamma^q|\beta\mathcal{H}) = e^{-\beta h(\psi_q)|C(\Gamma^q)|}\Theta(\Gamma^q|\beta\mathcal{H}) \qquad \text{and}$$

$$Z_q(V|\beta\mathcal{H}) = e^{-\beta h(\psi_q)|V|}\Theta_q(V|\beta\mathcal{H}),$$ where we denoted $C(\Gamma^q) = \operatorname{supp}\Gamma^q \cup \operatorname{Int}\Gamma^q$. Thus

$$Z(\Gamma^q|\beta\mathcal{H}) = e^{-\beta\Phi(\Gamma^q)}\prod_m Z_m(\operatorname{Int}_m\Gamma^q|\beta\mathcal{H})$$

with Φ related to Pirogov-Sinai's Ψ [2, formula (2.41)] by $\Phi(\Gamma^q) = \Psi(\Gamma^q) + h(\psi_q)|\Gamma^q|$. We will also use the results concerning contour models, especially Proposition 2.3 from [2] , though we do not need the Proposition 2.5 about the boundary term of the parametric contour model partition function.

The main step in Pirogov-Sinai's description of the phase diagram at small temperatures is to replace the model with a Hamiltonian $\mathcal{H}$ by suitable contour models with parameters. Crucial in this respect is their Proposition 2.6 in [2] , which asserts the existence of a vector of τ-functionals $\hat{F} = (F_1, \dots F_r)$ and parameters $b_1, \dots b_r$ such that

$$\Theta(\Gamma^q|\beta\mathcal{H}) = e^{b_q V(\Gamma^q)}\mathcal{X}(\Gamma^q|F_q) \tag{1}$$

for each $\Gamma^q \in \mathcal{C}_q$, $q=1,2,\dots r$ and

$$b_q = \beta h(\Psi_q) - s(F_q) + \alpha \tag{2}$$

with α defined by

$$\min_{1\leqslant q\leqslant r} b_q = 0 \; .$$

They prove the statement by rewriting (1) in the form of the equation

$$\hat{F} = \beta\hat{\Psi} + \hat{T}(\hat{F},\beta\mathcal{H}) \tag{3}$$

and showing that the nonlinear operator $\hat{T}$ is a contraction in a suitably chosen complete metric space of vectors of contour functionals.

Our aim here is to replace Proposition 2.6 in a more constructive way avoiding the use of the nonlinear equation (3). To this end we <u>define</u> for each $b\geqslant 0$ the contour functional F_q^b by induction in $|C(\Gamma^q)|$ so that

$$\frac{Z(\Gamma^q|\beta\mathcal{H})}{e^{-\beta h(\Psi_q)|C(\Gamma^q)|}} = e^{bV(\Gamma^q)}\mathcal{X}(\Gamma^q|F_q^b) \tag{4}$$

(note that $\mathcal{X}(\Gamma^q|F_q^b) = e^{-F_q^b(\Gamma^q)}\sum_{\partial^q\subset\mathrm{Int}\Gamma^q} e^{-F_q^b(\partial^q)}$ with $F_q^b(\partial^q)$ defined in the preceding steps of the induction). An immediate consequence of this definition is the relation connecting the partition function of the model with the partition function of the contour model with parameter:

$$Z_q(V|\beta\mathcal{H}) = e^{-\beta h(\Psi_q)|V|}\mathcal{X}(V|F_q^b,b) \quad .$$

Observing that from (2) and the fact that $\min b_q = 0$ it follows that $\alpha = s(\mathcal{H})$, free energy of the Hamiltonian model, and noting that the free energy $s(F_q^b)$ exists (by a simple argument similar to the usual proof) even when the contour functional F_q^b is not a τ-functional with large τ, it is natural to introduce the parameters b_q by

$$b_q = \sup\{b|s(F_q^b) - \beta h(\Psi_q) + b \leqslant s(\mathcal{H})\}. \tag{5}$$

The importance of these particular values of parameters is seen from the following

<u>Proposition</u> Let $\mathcal{H} = \mathcal{H}_0 + \tilde{\mathcal{H}}$ with $\mathcal{H}_0$ satisfying the Peierls's stability condition with a constant $\rho>0$ and let b_q be given by (5) for each $q=1,\dots r$. Then for $\|\tilde{\mathcal{H}}\|$ small enough and β large enough, and for each $q=1,\dots r$ we have

(i) $F_q^{b_q}$ are τ-functionals with τ large enough

(ii) $b_q = \beta h(\Psi_q) - s(F_q^{b_q}) + s(\mathcal{H})$

(iii) $\min_{1\leqslant m\leqslant r} b_m = 0$

(iv) b_q and $s(F^b_q q)$ are Lipschitz functions of $\beta\mathcal{H}$ with a small Lipschitz constant.

To give a hint of the proof we first introduce for each $q=1,\dots r$ and each $b\geqslant 0$ the τ-functional $\overline{F}^b_q$ by

$$\overline{F}^b_q(\Gamma^q) = \max(F^b_q(\Gamma^q), \tau|\Gamma^q|).$$

Defining now

$$\overline{b}_q = \sup\{b\,|\,s(\overline{F}^b_q) - \beta h(\Psi_q) + b \leqslant s(\mathcal{H})\},$$

we shall prove that $F^{\overline{b}}_q q$ is a τ-functional (i.e. that $\overline{F}^{\overline{b}}_q q = F^{\overline{b}}_q q$) and note that after showing this it is not difficult to show that $\overline{b}_q = b_q$ and thus that $F^b_q q$ is a τ-functional.

Thus we shall show by induction in $|C(\Gamma)|$ that

$$\frac{\mathcal{Z}(\Gamma|F^{\overline{b}}_q q)}{\mathcal{Z}(\mathrm{Int}\,\Gamma|F^{\overline{b}}_q q)} \leqslant e^{-\tau|\Gamma|} \tag{6}$$

(we skipped the superscript in Γ^q for simplicity). Using first (4) and observing that

$$Z_m(V|\beta\mathcal{H}) = e^{-\beta h(\Psi_m)|V|}\mathcal{Z}(V|F^b_m, b) \leqslant e^{(-\beta h(\Psi_m)+b)|V|}\mathcal{Z}(V|F^b_m)$$

we get

$$\mathcal{Z}(\Gamma|F^{\overline{b}}_q q) = e^{-\overline{b}_q V(\Gamma)-\beta\Phi(\Gamma)+\beta h(\Psi_q)V(\Gamma)}\prod_m Z_m(\mathrm{Int}_m\Gamma|\beta\mathcal{H}) \leqslant$$
$$\leqslant e^{-\beta\Phi(\Gamma)-\overline{b}_q V(\Gamma)+\beta h(\Psi_q)V(\Gamma)}\prod_m e^{(-\beta h(\Psi_m)+\overline{b}_m)V_m(\Gamma)}\mathcal{Z}(\mathrm{Int}_m\Gamma|F^{\overline{b}}_m m)$$

Noting further that

$$\mathcal{Z}(\mathrm{Int}_m\Gamma|F^{\overline{b}}_m m) = \mathcal{Z}(\mathrm{Int}_m\Gamma|\overline{F}^{\overline{b}}_m m)$$

by the induction hypothesis;

that

$$\mathcal{Z}(\mathrm{Int}_m\Gamma|\overline{F}^{\overline{b}}_m m) = e^{s(\overline{F}^{\overline{b}}_m m)V_m(\Gamma) + \delta(\tau)|\partial\mathrm{Int}_m\Gamma|}$$

with $|\delta(\tau)| \leqslant e^{-\tau}$ since $\overline{F}^{\overline{b}}_m m$ is a τ-functional;

and finally that

$$-\beta h(\Psi_m) + \overline{b}_m + s(\overline{F}^{\overline{b}}_m m) = s(\mathcal{H}) = -\beta h(\Psi_q) + \overline{b}_q + s(\overline{F}^{\overline{b}}_q q),$$

we get

$$\mathcal{Z}(\Gamma|F^{\overline{b}}_q q) \leqslant$$
$$\leqslant e^{-\beta\Phi(\Gamma)-\overline{b}_q V(\Gamma)+\beta h(\Psi_q)V(\Gamma)}\prod_m e^{(-\beta h(\Psi_m)+\overline{b}_m)V_m(\Gamma)}\mathcal{Z}(\mathrm{Int}_m\Gamma|\overline{F}^{\overline{b}}_m m) =$$
$$= e^{-\beta\Phi(\Gamma)-\overline{b}_q V(\Gamma)+\beta h(\Psi_q)V(\Gamma)+\delta(\tau)|\Gamma|}\prod_m e^{(-\beta h(\Psi_m)+\overline{b}_m+s(\overline{F}^{\overline{b}}_m m))V_m(}$$

$$= e^{-\beta\phi(\Gamma)+\mathcal{T}(\tau)|\Gamma|+s(\bar{F}^{b}_{q}q)V(\Gamma)} \leq e^{-\beta\phi(\Gamma)+2e^{-\tau}|\Gamma|}\mathcal{Z}(\mathrm{Int}\Gamma|\bar{F}^{b}_{q}q) =$$

$$= e^{-\beta\phi(\Gamma)+2e^{-\tau}|\Gamma|}\mathcal{Z}(\mathrm{Int}\Gamma|F^{b}_{q}q)$$

with the last equality following again from the induction hypothesis. This, for β large enough, implies (6).

To prove (iii) one observes that if $\min b_m$ were positive, then there would exist $\varepsilon > 0$ and parameters $\tilde{b}_q$ such that

$$s(\bar{F}^{\tilde{b}}_{q}q) - \beta h(\Psi_q) + \tilde{b}_q = s(\mathcal{H}) - \varepsilon \text{ for each } q=1,\dots\ r. \tag{7}$$

Noting that $F^{\tilde{b}}_{q}q$ is again a τ-functional (the proof is the same as above taking into account that one uses (7) in the form

$$s(\bar{F}^{\tilde{b}}_{q}q) - \beta h(\Psi_q) + \tilde{b}_q = s(\bar{F}^{\tilde{b}}_{m}m) - \beta h(\Psi_m) + \tilde{b}_m \qquad)$$

one gets

$$s(F^{\tilde{b}}_{q}q) - \beta h(\Psi_q) + \tilde{b}_q < s(\mathcal{H})$$

in the contradiction to

$$s(\mathcal{H}) \leq s(F^{\tilde{b}}_{q}q) - \beta h(\Psi_q) + \tilde{b}_q$$

following from

$$Z_q(V|\beta\mathcal{H}) \leq e^{(-\beta h(\Psi_q)+\tilde{b}_q)|V|}\mathcal{Z}(V|F^{\tilde{b}}_{q}q) .$$

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