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NOTE ON CANONIZING ORDERING THEOREMS FOR HALES JEWETT STRUCTURES

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This note presents a continuation of the research at the Winterschool 1982, see [1] .

Definition

Let $(A,\leq)$ be a finite totally ordered set. A lexicographic tree is a set T of intervals of $(A,\leq)$ satisfying the following rules:

(L1) $A \in T$

(L2) for all intervals I and J in T it follows that $I \cap J = \emptyset$ or $I \subseteq J$ or $J \subseteq I$.

(L3) for every interval I in T which contains at least two elements there exist mutually disjoint subintervals $I_0,\ldots,I_\ell$ in T, where $\ell \geq 1$, such that

$$I = I_0 \cup \ldots \cup I_\ell \ .$$

Convention

Let T be a lexicographic tree on $(A,\leq)$. For every interval $I \in T$ let $\mathrm{i\,Succ}(I) \subseteq T$ denote the set of immediate successors of I .

Denote by $T^* = \{I \in T \,;\, \mathrm{i\,Succ}(I) \neq \emptyset\}$.

Definition

(1) A quasi-ordering $\leq_q$ is a transitive relation such that each two elements are comparable, but possibly $I <_q J$ and $J <_q I$ for different elements I and J.

(2) Let T be a lexicographic tree on $(A,\leq)$. We say that a quasi-ordering $\leq_q$ on T^* extends the tree T iff $J \subseteq I$ implies that $I <_q J$, but not $J <_q I$.

Remark

If $\leq_q$ is a quasi-ordering on T^*, then $\leq_q$ induces an equivalence relation $\approx_q$ on T^* by $I \approx_q J$ iff $I \leq_q J$ and $J \leq_q I$. The quasi-ordering $\leq_q$ acts as a total ordering on the equivalence classes.

Notation

If $\leq_q$ is a quasi-ordering on T^*, let $T(0), T(1), T(2), \ldots$ be a monotonous (with respect to $\leq_q$) enumeration of the equivalence classes of $\approx_q$. Thus $T(0)$ is the least equivalence class and so forth. Recall that $T(0) = A$.

Definition (ordering-scheme)

Let A be a finite set. A 3-tuple $F = (\leq, T, \leq_q)$ is an ordering scheme for A iff

(1) $\leq$ is a total ordering on A ,

(2) T is a lexicographic tree for $(A,\leq)$,

(3) $\leq_q$ is a quasi-ordering on T^* which extends the tree T .

Next we show how an ordering scheme F on A can be used in order to define a total ordering on the set A^m of m-tuples over the set A .

Notation

Let $\vec{x} \in A^m$ and let X be a set of subsets of A . Then $\vec{x}\rceil X$ denotes the maximal subword of $\vec{x}$ consisting of all entries x_ν which belong to some member of X .

Notation

Let $\vec{x} = (x_0,\ldots,x_{m-1}) \in A^m$ and let $P = \{I_0,\ldots,I_{\ell-1}\}$ be a partition of A into mutually disjoint and nonempty subsets. Then

$$\vec{x}/_{P} = (\hat{x}_0,\ldots,\hat{x}_{m-1}) \in P^m$$

denotes the factorization of $\vec{x}$ with respect to P , i.e. $\hat{x}_\nu = I$ iff $x_\nu \in I$ for every $I \in P$ and $\nu \in \{0,\ldots,m-1\}$.

Notation

(1) if $\leq_q$ is a quasi - ordering extending the lexicographic tree T then

$$\mathrm{i\,Succ}(T(s)) = \cup \{\mathrm{i\,Succ}(I) \mid I \in T(s)\}$$

denotes the union of the sets of immediate successors of intervals in $T(s)$. As the intervals in T do not overlap (cf. (L2)) the ordering $\leq$ on A can be extended to $\mathrm{i\,Succ}(T(s))$.

(2) By $<_L$ we denote the lexicographic ordering on $(\mathrm{i\,Succ}(T(s)))^m$ with respect to the extension of $\leq$ to $\mathrm{i\,Succ}(T(s))$.

Definition

Let A be a finite set and let $F = (\leq, T, \leq_q)$ be an ordering scheme for A . The ordering $\leq_F$ on A^m , where m is a positive integer, is defined in the following way:

Let $\vec{x}$ and $\vec{y}$ be two different m-tuples in A^m .

$$\vec{x} \leq_F \vec{y} \text{ iff}$$

there exists a nonnegative integer s such that

$$\vec{x} \upharpoonright T(s)/_{\mathrm{i\,Succ}(T(s))} <_L \vec{y} \upharpoonright T(s)/_{\mathrm{i\,Succ}(T(s))}$$

and for every $i < s$ it follows that

$$\vec{x} \upharpoonright T(i)/_{\mathrm{i\,Succ}(T(i))} = \vec{y} \upharpoonright T(i)/_{\mathrm{i\,Succ}(T(i))} .$$

$\leq_F$ is also called a canonical ordering of A^m (induced by the ordering scheme F) .

We have the following results which solve the problem of the canonical set of orderingsof cubes stated in [1] :

Theorem 1 (irredundancy of $\leq_F$) :
Let A be a finite set. Then there exists a positive integer $m = m(A)$ with the following property:

For every pair F and F' of different ordering schemata for A there exist m-tuples $\vec{x}$ and $\vec{y}$ in A^m such that

$$\vec{x} <_F \vec{y} \quad \text{and} \quad \vec{y} <_{F'} \vec{x}$$

Theorem 2 (necessity of $\leq_F$) :
Let F be an ordering scheme for A . Let $m \leq n$ be positive integers. Then there exists an ordering $\leq$ on A^n such that for every $f \in [A]\binom{n}{m}$ holds:

$$\vec{x} \leq_F \vec{y} \quad \text{iff} \quad f \cdot \vec{x} \leq f \cdot \vec{y}$$

for all m-tuples $\vec{x}$ and $\vec{y}$ in A^m .

Theorem 3 (sufficiency of $\leq_F$) :
Let A be a finite set. For every nonnegative integer m there exists a nonnegative integer n with the following property:

for every totoal ordering on A^n there exists an m-parameter word $f \in [A]\binom{n}{m}$ and there exists an ordering scheme F for A such that

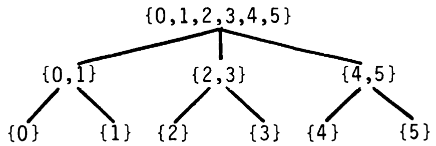
$$\vec{x} \leq_F \vec{y} \quad \text{iff} \quad f \cdot \vec{x} \leq f \cdot \vec{y}$$

for all m-tuples $\vec{x}$ and $\vec{y}$ in A^m .

It is interesting to note that the relatively complicated structure of unavoidable orderings of A^n appears only for larger alphabets A . For small alphabets the canonical set of orderings was determined earlier in [1] and it represents simpler results.
The first non-trivial example appears at $A = \{0,1,2,3,4,5\}$ and can be depicted

as follows:

T :



$\leq$: $0 < 1 < 2 < 3 < 4 < 5$

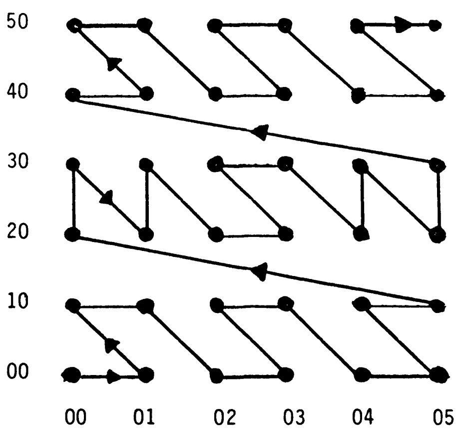
$\leq_q$: $\{0,1\} \approx \{4,5\} <_q \{2,3\}$

(all other pairs are determined as $<_q$ is an extension of T) .

Consequently: $T(0) = \{0,1,2,3,4,5\}$

$T(1) = \{0,1,4,5\}$, $T(2) = \{2,3\}$.

The structure of the standard ordering $\leq_F$ may be indicated by the following pairs of A^2 :



The proofs of the above theorems will appear elsewhere.

References

[1] J. Nesetril, H.J. Prömel, V. Rödl and B. Voigt: A canonical ordering theorem, a first attempt. In: "Proceedings of the 10th Winterschool", Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II numero 2(1982), 193 - 197 .

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