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In: Zdeněk Frolík (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 3. pp. [197]--200.

Persistent URL: http://dml.cz/dmlcz/701311

AN ELEMENTARY APPROACH TO SOME APPLICATIONS OF NONSTANDARD ANALYSIS

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It is possible in many cases to prove various assertions by means of the methods of nonstandard analysis without complicate logical means. Moreover, it is possible to avoid the ultrapover construction. Such approaches are known (see e.g. [2] , [4]). The aim of this paper is to show that such an approach is possible also in the uniform spaces, in particular in topological vector spaces. To illustrate such a situation we present simple proofs of two known theorems.

Notations and notions.

Let X be a non empty set, I be an infinite index set. Let X^I be the set of all functions $x = (x_n)_{n \in I}$ from I to X. We identify two functions $x, y \in X^I$, if $x_n = y_n$ for "almost all" n. This means that the set $\{n \in I; x_n = y_n\}$ belongs to a fixed ultrafilter F of subsets of I containing all complements of finite subsets (so called nontrivial ultrafilter). Recall that F is ultrafilter, if 1) $\emptyset \notin F$ 2) $A \cap B \in F$ for all $A, B \in F$ 3) $B \in F$, if $B \supset A$ for some $B \in F$ and 4) either $A \in F$ or $I - A \in F$ for every $A \subset I$.

Hence, if we define $(x_n)_n \sim (y_n)_n$ iff $\{n \in I; x_n = y_n\} \in F$, we obtain an equivalence relation. We shall denote by ${}^*X$ the corresponding factor space and by $[(x_n)_n]$ the member of ${}^*X$ containing $(x_n)_n$. If $x \in X$, then $[(x)_n] \in {}^*X$, hence $X \subset {}^*X$ in some sense.

Similarly we can define ${}^*A$ for all subsets $A \subset X$ by ${}^*A = \{[(x_n)_n] ; x_n \in A$ for all $n \in I\}$. Of course, in a class $[(x_n)_n]$ there exist also functions $(y_n)_n$ with some members out of A, but in any case $\{n ; y_n \in A\} \in F$.

Finally for a function $f: A \to B$ we can define its extension ${}^*f: {}^*A \to {}^*B$ by the formula ${}^*f([(x_n)_n]) = [(f(x_n))_n]$. It is not difficult to prove that the definition is correct, i.e. it does not depend on the choice of $(x_n)_n$.

If we want to do some analytical considerations, we need some further structures. Suppose for example that X is a topological space. Then we can define for every $x \in X$ the monade $m(x) = \bigcap \{ {}^{*}U;\ U \text{ is open}, x \in U \}$. If, moreover X is a metric space with the metric d, we can define a relation $\approx$ on ${}^{*}X$ by the following way: $x \approx y$ iff $\{ n \in I;\ d(x_n, y_n) < r \} \in F$ for every $r > 0$. A similar relation can be defined in any uniform space, too.

Uniform spaces.

Let (X,U) be a uniform space with the (infinite) uniformity U (see e.g. [3]). As an index set we take $I = U$ and we shall work with a fixed ultrafilter F of subsets of U, containing all sets of the form $\{ u \in U;\ u \subset v \}$, where $v \in U$. If $x = [(x_u)_u]$, $y = [(y_u)_u]$, then we write $x \approx y$ iff $\{ v \in U;\ (x_v, y_v) \in u \} \in F$ for every $u \in U$.

The proofs of the following propositions are quite simple and therefore we shall omit them.

Proposition 1. $m(x) = \{ y \in {}^{*}X;\ y \approx x \}$ for every $x \in X$.

Proposition 2. A set $A \subset X$ is open iff $m(x) \subset {}^{*}A$ for every $x \in A$.

Proposition 3. If X is Hausdorff, then $m(x) \cap m(y) = \emptyset$ whenever $x, y \in X$, $x \neq y$.

Theorem 1. Let C, Y be two uniform spaces, C be compact. If $f: C \to Y$ is continuous, then f is uniformly continuous.

Proof. 1. If $r \in C$, $x \in {}^{*}C$, then $f(r) \approx {}^{*}f(x)$.

Indeed, since f is continuous in r, to every open neighbourhood G of f(r) there is an open neighbourhood H of r such that $f(H) \subset G$. Let $x = [(x_u)_u]$. Since $x \approx r$, we have $\{ u \in U;\ x_u \in H \} \in F$, hence $\{ u \in U;\ f(x_u) \in G \} \in F$, too. We obtain ${}^{*}f(x) \in {}^{*}G$ for every G, hence ${}^{*}f(x) \in m(f(r))$. Therefore ${}^{*}f(x) \approx f(r)$ by Prop. 1.

2. To every $x \in {}^{*}C$ there exists $r \in C$ such that $x \in m(r)$.

Indirectly, let there exist $x \in {}^{*}C$ such that $x \notin m(r)$ for every $r \in C$. Then to every $r \in C$ there exists an open neighbourhood G_r of r such that $x \notin {}^{*}G_r$. Since $\{ G_r;\ r \in C \}$ covers C and C is compact, there are $r_1, \dots, r_k \in C$ such that $C \subset G_{r_1} \cup \dots \cup G_{r_k}$. Therefore $x \in {}^{*}C \subset {}^{*}G_{r_1} \cup \dots \cup {}^{*}G_{r_k}$ and there is i with $x \in {}^{*}G_{r_i}$, what is a contradiction.

3. If $x, y \in {}^{*}C$, $x \approx y$, then ${}^{*}f(x) \approx {}^{*}f(y)$.

By Part 2 there are $r, s \in C$ such that $x \in m(r)$, $y \in m(s)$. Then $x \approx r$, $y \approx s$ by Prop. 1, hence ${}^{*}f(x) \approx f(r)$, ${}^{*}f(y) \approx f(s)$ by Part 1. Evidently $r \approx s$, hence $f(r) \approx f(s)$ and therefore ${}^{*}f(x) \approx {}^{*}f(y)$.

4. f is uniformly continuous.

Indirectly, let there exist $v \in U$ such that for every $u \in U$ there are $x_u, y_u \in C$ with $(x_u, y_u) \in u$ and $(f(x_u), f(y_u)) \notin v$. Put $x = [(x_u)_u]$, $y = [(y_u)_u]$. Then $x, y \in {}^*C$, $x \approx y$, but ${}^*f(x) \approx {}^*f(y)$ does not hold, what is a contradiction to Part 3.

Topological vector spaces

The proof of the following theorem is similar to a nonstandard one given in [1]. In fact we formulate and prove a more general result. Note first that a uniformity U is said to be invariant, if for any $u \in U$ and any $x, y \in X$ we have $(x,y) \in U$ iff $(x + z, y + z) \in U$.

Theorem 2. Let X be a finite dimensional vector space. Then there exists unique Hausdorff topology G on X such that

(i) G is induced by an invariant uniformity

(ii) for any $x \in X$ the function $a \longmapsto ax$ is as a function of the real variable continuous at 0

(iii) $m(0)$ is a vector subspace over ${}^*R$ of ${}^*X$.

Proof. Suppose U is invariant such that (i), (ii), (iii) are satisfied. We prove that $m(0)$ consists of all linear combinations $a_1x_1 + a_2x_2 + \dots + a_nx_n$, where a_i $(i=1,2,\dots,n)$ are infinitesimals in ${}^*R$ and $x_1, x_2, \dots, x_n$ is a basis in X. Suppose the last is not true. Then $x = a_1x_1 + \dots + a_nx_n$, where at least one of a_i $(i=1,2,\dots,n)$ is not infinitesimal. With no loss of generality we may suppose all a_i to be finite. If not we can choose a member K that

$$x = K\ ((a_1/K)\ x_1 + \dots + (a_n/K)x_n)$$

and all a_i/K are finite. Then we can take $(a_1/K)\ x_1 + \dots + (a_n/K)x_n$ instead of x.

So let $a_1, \dots, a_j$ $(j \leqq n)$ be finite and not infinitesimal. Then $a_i = b_i + c_i$, where $b_i \in R$ and c_i are infinitesimal for $i = 1, \dots, j$. Thus $x = b + c$, where $b \in X$, $b \neq 0$ and c infinitesimal. Thus $x \in m(b)$ and we have $m(b) \cap m(0) \neq \emptyset$. It is a contradiction to Prop. 3. So for any uniformity which satisfies (i), (ii), (iii) the monad $m(0)$ is the same.

Now it remains to prove that if U and V are two invariant uniformities on X with $m_U(0) = m_V(0)$, the topologies induced by U and V coincide. In fact, we obtain without difficulties $m_U(x) = m_V(x)$ for any $x \in X$. If the topologies U and V were not identical, then there is a set G open in the first and not in the second or conversly. Suppose the first case. Then by Prop. 2 we have $m_U(x) \subset {}^*G$ for any $x \in G$, but $m_V(x_o) \not\subset {}^*G$ for some $x_o \in G$. It is a contradiction.

Corollary 1. Let X be a finite dimensional vector space. Then there exists unique Hausdorff topology on X such that X is a topological space.

Proof. Any Hausdorff topology in which X is a topological vector space evidently satisfies (i) and (ii). The property (iii) follows from the continuity of the operations of the sum and multiplication.

Corollary 2. Any Hausdorff topology satisfying (i), (ii), (iii) on a finite dimensional vector space performs the space into a Hausdorff topological space.

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