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FLOWS IN INFINITE NETWORKS

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<u>Introduction</u>. The main result of this note may be viewed as a generalized version of Gale's celebrated feasibility theorem on flows in networks [1, p.38]. In our general context, flows will be certain biadditive set functions $\nu: \Sigma \times \Sigma \to X$, where Σ is some algebra of sets and X is a Dedekind complete ordered vector space. In the classical situation, Σ is just the whole power set $\mathcal{P}(S)$ for some finite set S of nodes, while X is the real line $\mathbb{R}$. There is, however, a considerable demand for a more general setting: For instance, infinite sets of agents arise quite naturally in modern Mathematical Economics, infinite commodity sets do occur as soon as one is interested in the dynamic behaviour of a given system, and last, not least one wants to handle the case of multi-commodity flows. The present approach to flows in infinite networks is completely different from the various known proofs in the finite case. Here, we shall make essential use of sublinear operators and the interpolation theorem due to Mazur-Orlicz. These techniques are close in spirit to those of Fuchssteiner [2;3] and König-Neumann [6]. There are, however, some significant differences and simplifications, since neither disintegration tools nor localized order structures will be used here. A similar approach has been employed in [7] to produce a Ford-Fulkerson type theorem concerning maximal flows and minimal cuts in infinite networks. The following section contains the main result of the present paper. In the remaining sections we discuss some immediate applications from which our interest in generalized networks actually arose.

<u>An Extended Flow Theorem</u>. In the following, let S be a non-empty set endowed with some algebra Σ of subsets. X stands for an ordered vector space which is assumed to be Dedekind complete in the sense that each upper bounded subset has a supremum. We add a smallest and a greatest element $\pm\infty$ to X and extend the algebraic operations

in the usual way, i.e. we define $0\cdot(\pm\infty) = 0$, $t\cdot(\pm\infty) = \pm\infty$ for all real $t > 0$, $x \pm \infty = \pm\infty$ for all $x \in X$. Finally let us fix a pair of biadditive set functions $\sigma,\tau : \Sigma \times \Sigma \to X$ such that

$$(1)\qquad \sigma(A,B) \leq \tau(A,B) \qquad \text{for all disjoint } A,B \in \Sigma$$

and a pair of additive set functions $\lambda : \Sigma \to X \cup \{-\infty\}$ and $\mu : \Sigma \to X \cup \{+\infty\}$ such that $\lambda \leq \mu$ on Σ. Then we have:

Theorem 1. *There exists a biadditive mapping* $\nu : \Sigma \times \Sigma \to X$ *such that:*

$$(2)\qquad \sigma(A,B) \leq \nu(A,B) \leq \tau(A,B) \qquad \forall A,B \in \Sigma \text{ with } A \cap B = \emptyset$$

$$(3)\qquad \lambda(A) \leq \nu(A,S) - \nu(S,A) \leq \mu(A) \qquad \forall A \in \Sigma$$

if and only if the following condition is satisfied:

$$(4)\qquad \lambda(A),\ -\mu(\bar{A}) \leq \tau(A,\bar{A}) - \sigma(\bar{A},A) \qquad \forall A \in \Sigma$$

where $\bar{A} = S \setminus A$ *denotes the complement.*

Proof. The necessity of condition (4) is obvious; so we only have to prove the sufficiency. According to the transformations

$$\tilde{\sigma} = 0,\ \tilde{\tau} = \tau - \sigma,\ \tilde{\nu} = \nu - \sigma$$
$$\tilde{\lambda} = \lambda - \sigma(\cdot,S) + \sigma(S,\cdot),\ \tilde{\mu} = \mu - \sigma(\cdot,S) + \sigma(S,\cdot),$$

we may assume that $\sigma = 0$. Our basic assumption (1) then reads $\tau(A,B) \geq 0$ for all disjoint $A,B \in \Sigma$. We now establish an appropriate functional-analytic setting for our problem. Let E denote the space of all Σ-measurable simple functions $\varphi : S \to \mathbb{R}$, and let F consist of all $\Sigma\circ\Sigma$-measurable simple functions $\varphi : S \times S \to \mathbb{R}$, where $\Sigma \circ \Sigma$ is the algebra on $S \times S$ generated by $\Sigma \times \Sigma$. Our theory will be dominated by the sublinear operator $\theta : E \to F$ given by

$$\theta(\varphi)(u,v) := \max\{\varphi(u) - \varphi(v),\, 0\} \qquad \forall u,v \in S,\ \varphi \in E.$$

The following properties of θ can be easily checked. As usual, χ_A stands for the characteristic function of a given set A.

$$(5)\qquad \theta(\chi_A) = \chi_{A\times\bar{A}},\quad \theta(-\chi_A) = \chi_{\bar{A}\times A} \qquad \forall A \in \Sigma.$$

$$(6)\qquad \theta(\varphi + t) = \theta(\varphi) \qquad \forall \varphi \in E,\ t \in \mathbb{R}.$$

$$(7)\qquad \theta(\varphi - \psi) = \theta(\varphi) + \theta(-\psi) \qquad \forall \varphi,\psi \in E_+ \text{ with } \varphi\psi = 0.$$

$$(8)\qquad \theta\Big(\sum_{i=1}^{n} t_i\, \chi_{A_i}\Big) = \sum_{i=1}^{n} t_i\, \theta(\chi_{A_i}) \qquad \forall t_1,\dots,t_n \geq 0$$
$$\text{and } A_1,\dots,A_n \in \Sigma \text{ with } A_1 \supset \dots \supset A_n.$$

Thus, in a precise sense, the operator θ behaves almost linearly. Next, let $\hat{\tau} : \Sigma \circ \Sigma \to X$ denote the additive mapping corresponding to τ via the formula

$$\hat{\tau}\left(\bigcup_{i=1}^{n} A_i \times B_i\right) = \sum_{i=1}^{n} \tau(A_i, B_i)$$

for every finite system of pairwise disjoint rectangles $A_i \times B_i \in \Sigma \times \Sigma$. Using an elementary notion of an X-valued integral, we may define

$$\rho(\psi) := \int_{S\times S} \theta(\psi(u,v))(u,v)\,d\hat{\tau}(u,v) \qquad \forall \psi \in G,$$

where G denotes the space of all $\Sigma\circ\Sigma$-measurable simple functions $\psi : S \times S \to E$. If $\tau \geq 0$ on $\Sigma \times \Sigma$ and hence $\hat{\tau} \geq 0$ on $\Sigma \circ \Sigma$, then $\rho : G \to X$ is certainly a sublinear operator. But a more careful calculation shows that ρ is sublinear even under the present weaker assumption on τ. Finally, let K_λ and K_μ consist of all $\varphi \in E_+$ which are integrable with respect to λ and μ, respectively. Now, let $\varphi \in K_\lambda$ and $\psi \in K_\mu$ be arbitrarily given. We put $\tilde{\varphi} := (\varphi - \psi)^+$ and $\tilde{\psi} := (\varphi - \psi)^-$ so that $\varphi - \psi = \tilde{\varphi} - \tilde{\psi}$, $\tilde{\varphi}\tilde{\psi} = 0$, $\tilde{\varphi} \in K_\lambda$, $\tilde{\psi} \in K_\mu$, and $\varphi - \tilde{\varphi} = \psi - \tilde{\psi} \in K_\lambda \cap K_\mu$. From $\lambda \leq \mu$ we conclude

$$\int_S \varphi\,d\lambda - \int_S \psi\,d\mu \leq \int_S \tilde{\varphi}\,d\lambda - \int_S \tilde{\psi}\,d\mu.$$

We now consider representations of the type

$$\tilde{\varphi} = \sum_{i=1}^{n} s_i \chi_{A_i} \qquad \text{and} \qquad \tilde{\psi} = \sum_{j=1}^{m} t_j \chi_{B_j},$$

where $s_i, t_j \geq 0$ and $A_i, B_j \in \Sigma$ with $\lambda(A_i), \mu(B_j) \in X$ such that $A_1 \supset \ldots \supset A_n$, $B_1 \subset \ldots \subset B_m$, and $A_1 \cap B_m = \emptyset$. From the properties (5) - (8) of θ and from our assumption (4) for the case $\sigma = 0$ we conclude that

$$\begin{aligned}
\int_{S\times S} \theta(\varphi - \psi)\,d\hat{\tau} &= \int_{S\times S} \theta(\tilde{\varphi} - \tilde{\psi})\,d\hat{\tau} \\
&= \int_{S\times S} (\theta(\tilde{\varphi}) + \theta(-\tilde{\psi}))\,d\hat{\tau} = \int_{S\times S} \theta(\tilde{\varphi})\,d\hat{\tau} + \int_{S\times S} \theta(t_1 + \ldots + t_m - \tilde{\psi})\,d\hat{\tau} \\
&= \sum_{i=1}^{n} s_i \tau(A_i, \bar{A}_i) + \sum_{j=1}^{m} t_j \tau(\bar{B}_j, B_j) \\
&\geq \sum_{i=1}^{n} s_i \lambda(A_i) - \sum_{j=1}^{m} t_j \mu(B_j) = \int_S \tilde{\varphi}\,d\lambda - \int_S \tilde{\psi}\,d\mu.
\end{aligned}$$

Identifying E with the space of all constant functions in G, we thus arrive at the estimates

$$\rho(\varphi - \psi) \geq \int_S \varphi\,d\lambda - \int_S \psi\,d\mu \qquad \forall \varphi \in K_\lambda, \psi \in K_\mu.$$

Hence a vector-valued version of the Mazur-Orlicz theorem supplies us with a linear mapping $\nu : G \to X$ satisfying $\xi \leq \rho$ on G as well as

$$\int_S \varphi d\lambda \leq \xi(\varphi) \qquad \forall \varphi \in K_\lambda \quad , \quad \int_S \varphi d\mu \geq \xi(\varphi) \qquad \forall \varphi \in K_\mu ,$$

see for example [8, p.79]. By means of this mapping, we define

$$\nu(A,B) := \xi(\psi_{A,B}) \qquad \forall A,B \in \Sigma,$$

where $\psi_{A,B} \in G$ denotes the function being constant to χ_A on $S \times B$ and to 0 on $S \times \overline{B}$. The set function $\nu : \Sigma \times \Sigma \to X$ is certainly biadditive. Moreover, from $\xi \leq \rho$ on G one easily deduces that

$$-\tau(\overline{A}, A \cap B) \leq \nu(A,B) \leq \tau(A, \overline{A} \cap B) \qquad \forall A,B \in \Sigma.$$

This implies $0 \leq \nu(A,B) \leq \tau(A,B)$ for all disjoint $A,B \in \Sigma$ as well as $\nu(S,A) = 0$ for all $A \in \Sigma$. Finally, given any $A \in \Sigma$ such that $\lambda(A) \in X$ resp. $\mu(A) \in X$, we obtain

$$\lambda(A) \leq \xi(\psi_{A,S}) = \nu(A,S) = \nu(A,S) - \nu(S,A),$$
$$\mu(A) \geq \xi(\psi_{A,S}) = \nu(A,S) = \nu(A,S) - \nu(S,A),$$

respectively. Note that the resulting estimates are obvious if $\lambda(A) = -\infty$, resp. $\mu(A) = +\infty$. Thus ν has the desired properties.

<u>Corollary 2</u>. <u>Assume that</u> $\sigma \leq \tau$ <u>holds on</u> $\Sigma \times \Sigma$. <u>Then there exists a biadditive set function</u> $\nu : \Sigma \times \Sigma \to X$ <u>satisfying</u> $\sigma \leq \nu \leq \tau$ <u>on</u> $\Sigma \times \Sigma$ <u>and</u> (3) <u>if and only if condition</u> (4) <u>is fulfilled</u>.

<u>Proof</u>. It suffices again to show the sufficiency for the case $\sigma = 0$. Thus we suppose that $\tau \geq 0$ on $\Sigma \times \Sigma$. Hence the mapping ν from the preceding proof satisfies $\nu \leq \tau$ on $\Sigma \times \Sigma$ and $\nu(A,B) \geq 0$ for all disjoint $A,B \in \Sigma$. Let $\hat{\nu} : \Sigma \circ \Sigma \to X$ denote the corresponding additive set function, and consider its positive part given by

$$\gamma(V) := \sup\{\hat{\nu}(U) : U \in \Sigma \circ \Sigma \text{ with } U \subset V\} \in X \quad \forall\, V \in \Sigma \circ \Sigma.$$

Obviously, $\gamma : \Sigma \circ \Sigma \to X$ is well-defined and additive. And it is easily verified that $0 \leq \gamma \leq \hat{\tau}$ on $\Sigma \circ \Sigma$ as well as $\gamma(A \times \overline{A}) = \nu(A, \overline{A})$ for all $A \in \Sigma$. Hence the definition $\tilde{\nu}(A,B) := \gamma(A \times B)$ for all $A,B \in \Sigma$ yields a biadditive mapping $\tilde{\nu} : \Sigma \times \Sigma \to X$ with the desired properties.

Let us state an immediate consequence: There exists a biadditive mapping $\nu : \Sigma \times \Sigma \to X$ satisfying $\sigma \leq \nu \leq \tau$ and (3) if and only if there is a pair of biadditive mappings $\nu_1, \nu_2 : \Sigma \times \Sigma \to X$ satisfying

$\sigma \leq \nu_1, \nu_2 \leq \tau$ as well as $\lambda(A) \leq \nu_1(A,S) - \nu_1(S,A)$ and $\nu_2(A,S) - \nu_2(S,A) \leq \mu(A)$ for all $A \in \Sigma$. A similar characterization holds in the situation of theorem 1.

We finally note that the preceding result admits an important measure theoretic interpretation. Let us assume $X = \mathbb{R}$, for simplicity, and consider a pair of finite measures $\tilde{\sigma}, \tilde{\tau} : \Sigma \otimes \Sigma \to \mathbb{R}$ with $\tilde{\sigma} \leq \tilde{\tau}$, where $\Sigma \otimes \Sigma$ is the usual σ-algebra on $S \times S$ generated by $\Sigma \times \Sigma$. Then $\tilde{\sigma}$ and $\tilde{\tau}$ give rise to biadditive set functions σ and τ on $\Sigma \times \Sigma$ via $\sigma(A,B) = \tilde{\sigma}(A \times B)$ and $\tau(A,B) = \tilde{\tau}(A \times B)$ for all $A,B \in \Sigma$. Conversely, by standard measure theory, every biadditive mapping $\nu : \Sigma \times \Sigma \to \mathbb{R}$ satisfying $\sigma \leq \nu \leq \tau$ on $\Sigma \times \Sigma$ canonically induces a finite measure $\tilde{\nu} : \Sigma \otimes \Sigma \to \mathbb{R}$ such that $\tilde{\sigma} \leq \tilde{\nu} \leq \tilde{\tau}$ on $\Sigma \otimes \Sigma$ and $\nu(A,B) = \tilde{\nu}(A \times B)$ for all $A,B \in \Sigma$. Thus corollary 2 contains a feasibility theorem for finite measures on $\Sigma \otimes \Sigma$ as a special case.

Biadditive mappings with given marginals. Let S_1 and S_2 be sets endowed with algebras Σ_1 and Σ_2 of subsets, respectively, and consider biadditive set functions $\sigma, \tau : \Sigma_1 \times \Sigma_2 \to X$ such that $\sigma \leq \tau$. Moreover, for $i=1,2$ let $\lambda_i : \Sigma_i \to X \cup \{-\infty\}$ and $\mu_i : \Sigma_i \to X \cup \{+\infty\}$ be additive such that $\lambda_i \leq \mu_i$.

Theorem 3. *There exists a biadditive mapping* $\nu : \Sigma_1 \times \Sigma_2 \to X$ *such that*

$$\lambda_1 \leq \nu(\cdot, S_2) \leq \mu_1 \quad \text{and} \quad \lambda_2 \leq \nu(S_1, \cdot) \leq \mu_2$$

if and only if the following condition is fulfilled:

$$\lambda_1(A_1) - \mu_2(A_2),\ \lambda_2(\bar{A}_2) - \mu_1(\bar{A}_1) \leq \tau(A_1, \bar{A}_2) - \sigma(\bar{A}_1, A_2)$$

for all $A_i \in \Sigma_i$.

Proof. We may assume that $S_1 \cap S_2 = \emptyset$. Now endow $S := S_1 \cup S_2$ with the canonocal algebra Σ coming from Σ_1 and Σ_2 and consider the trivial extensions of σ and τ to $\Sigma \times \Sigma$ being 0 on $\Sigma_1 \times \Sigma_1$, $\Sigma_2 \times \Sigma_2$, $\Sigma_2 \times \Sigma_1$. Finally, let λ be $= \lambda_1$ on Σ_1 and $= -\mu_2$ on Σ_2, and define μ to be $= \mu_1$ on Σ_1 and $= -\lambda_2$ on Σ_2. Then the assertion can be easily deduced from corollary 2.

This theorem is closely related to results of Kellerer [4;5] concerning functions and measures on product spaces with given marginals, but the technicalities are rather different. The present approach avoids, for instance, the weak compactness of order intervals in L^1 and the Radon-Nikodym theorem. Of course, by the concluding remarks of the last section, theorem 3 immediately includes the case of finite measures and can also be extended to the case of σ-finite measures [4].

An Economic Application. In this section, we consider a triplet (P,Σ_P), (Q,Σ_Q), (R,Σ_R) of sets endowed with certain algebras of subsets. For the sake of interpretation, P will stand for the producers, Q for the consumers, and R for the goods of a given economic system. In modern economics, the emphasis lies on the coalitions the agents may form and on the submarkets which may be built up by the given goods. Thus Σ_P, Σ_Q, Σ_R rather than P,Q,R will be of decisive importance. Suppose that we are given biadditive set functions

$$\rho_1,\rho_2 : \Sigma_R \times \Sigma_P \to X \qquad \text{with } \rho_1 \leq \rho_2,$$
$$\sigma_1,\sigma_2 : \Sigma_R \times \Sigma_Q \to X \qquad \text{with } \sigma_1 \leq \sigma_2.$$

These mappings will serve as lower and upper raw-material bounds for the producers and as lower and upper saturation bounds for the consumers, respectively. We also assume that there are lower and upper supply and demand functions, i.e. additive mappings

$$a_1 : \Sigma_P \to X \cup \{-\infty\},\quad a_2 : \Sigma_P \to X \cup \{+\infty\} \text{ with } a_1 \leq a_2,$$
$$b_1 : \Sigma_Q \to X \cup \{-\infty\},\quad b_2 : \Sigma_Q \to X \cup \{+\infty\} \text{ with } b_1 \leq b_2.$$

Of course, in concrete applications some of these mappings will be identical to 0, $+\infty$, or $-\infty$, which will considerably simplify some of the subsequent formulae. Now, the obvious problem is to find a production and a consumption plan which are compatible with the given situation. To be more precise, we are looking for a pair of biadditive set functions $f : \Sigma_R \times \Sigma_P \to X$ and $g : \Sigma_R \times \Sigma_Q \to X$ such that the following conditions are satisfied:

$$\begin{aligned} &\rho_1 \leq f \leq \rho_2 \quad , \quad a_1 \leq f(R,\cdot) \leq a_2, \\ (9) \qquad &\sigma_1 \leq g \leq \sigma_2 \quad , \quad b_1 \leq g(R,\cdot) \leq b_2, \\ &g(\cdot,Q) \leq f(\cdot,P). \end{aligned}$$

This problem will not have any solution unless the given data fit together in a suitable way. In order to give the appropriate conditions, we introduce for each $A \in \Sigma_R$ the following notations:

$$\begin{aligned}
\pi(A) &:= \inf\{a_2(U) - \rho_1(\overline{A},U) + \rho_2(A,\overline{U}) : U \in \Sigma_P\} \\
&= [(a_2 - \rho_1(\overline{A},\cdot)) \wedge \rho_2(A,\cdot)](P) \\
\eta(A) &:= \sup\{b_1(V) - \sigma_2(\overline{A},V) + \sigma_1(A,\overline{V}) : V \in \Sigma_Q\} \\
&= [(b_1 - \sigma_2(\overline{A},\cdot)) \vee \sigma_1(A,\cdot)](Q) \\
\pi_o &:= \inf\{\rho_2(R,U) - a_1(U) : U \in \Sigma_P\} \\
&= [(\rho_2(R,\cdot) - a_1) \wedge 0](P) \\
\eta_o &:= \sup\{\sigma_1(R,V) - b_2(V) : V \in \Sigma_Q\} \\
&= [(\sigma_1(R,\cdot) - b_2) \vee 0](Q)
\end{aligned}$$

Here, $\pi(A)$ may be viewed as the maximal supply for the submarket $A \in \Sigma_R$, whereas $\eta(A)$ is just the minimal demand for this submarket. Indeed, every production plan f certainly satisfies $f(\cdot,P) \leq \pi$, whereas $\eta \leq g(\cdot,Q)$ holds for every distribution plan g. Hence the condition $\eta \leq \pi$ on Σ_R turns out to be necessary for the feasibility of our problem. A similar reasoning leads to the condition $\eta_o \leq \pi_o$ which actually means $\eta_o = \pi_o = 0$. We shall see that these conditions suffice. This generalizes a result of Fuchssteiner [3, p.67] concerning a more restrictive situation. The interested reader will find a lot of further information and background material in [3].

<u>Theorem 4</u>. <u>There exist biadditive mappings</u> f <u>and</u> g <u>satisfying the conditions stated in</u> (9) <u>if and only if</u> $\eta \leq \pi$ <u>on</u> Σ_R <u>and</u> $\eta_o = \pi_o = 0$.

<u>Proof</u>. We may assume that P,Q,R are pairwise disjoint and endow $S := P \cup Q \cup R$ with the canonical algebra Σ coming from Σ_P, Σ_Q, Σ_R. Further, let $\tau : \Sigma \times \Sigma \to X$ be given by ρ_2 on $\Sigma_R \times \Sigma_P$, by σ_2 on $\Sigma_Q \times \Sigma_R$ modulo an obvious change of variables, and by 0 on the remaining parts. The mapping $\sigma : \Sigma \times \Sigma \to X$ is similarly defined by means of ρ_1 and σ_1. Finally, the additive set functions λ and μ on Σ are given by

$$\begin{aligned}
&\lambda = -a_2 \text{ on } \Sigma_P, \quad \lambda = b_1 \text{ on } \Sigma_Q, \quad \lambda = 0 \text{ on } \Sigma_R, \\
&\mu = -a_1 \text{ on } \Sigma_P, \quad \mu = b_2 \text{ on } \Sigma_Q, \quad \mu = \infty \text{ on } \Sigma_R.
\end{aligned}$$

Then some elementary calculations reveal that the present conditions are exactly those of corollary 2. The desired set functions f and g are now obtained by the restriction of some flow ν to $\Sigma_R \times \Sigma_P$ and to $\Sigma_Q \times \Sigma_R$, respectively.

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