SYMMETRIC p-NORMED SPACES
FOR $0<p\leq 1$

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In this paper we introduce the notion of a symmetric p-normed space, $0<p\leq 1$, a natural extension of that of a symmetric normed space. (See [4]).

For these spaces we extend (usually without proofs) some results of [1], [2], [4]. For instance we prove that for a symmetric p-normed sequence space E such that the Boyd indices p_E and q_E are not trivial, the triangular projection T acts continuously on the corresponding symmetric p-normed space C_E and conversly, if T acts continuously on C_E then the Boyd indices of E are not trivial. Particularly the space $C_{q,p}$, for $0<p<1<q$ and $1 + 1/q>1/p$, has this property but C_p for $0<p<1$ has it not.

Another interesting result is that the spaces C_p, for $0<p<1$, are primary, obtaining thus an extension of a previous result of J. Arazy [2]. As a general remark we point out that the proofs follow the lines of these of the papers [1], [2] and [4].

§ 1 - General theory of symmetric p-normed spaces, $0<p\leq 1$.

As a general rule we use the terminology of [4] and [1].

First we introduce the notion of a symmetric p-norm.

Definition 1.1. Let's denote $B(\ell_2)$ the space of all linear bounded operators on ℓ_2. A positive function $|X|_s$ defined on an ideal C of $B(\ell_2)$ is called a symmetric p-norm if the following properties hold:

1) $|X|_s = 0$ if and only if $X = 0$.
2) $|\lambda X|_s = |\lambda| \cdot |X|_s$ for $X\in C$, $\lambda \in \mathbb{C}$.
3) (p-convexity property). For every two sequences $(\xi_{1j})_{j=1}^{\infty}$, $(\xi_{2j})_{j=1}^{\infty}$ of real numbers and for every orthonormal system $(\varphi_j)_{j=1}^{\infty}$ of elements of ℓ_2 the following inequality holds:

$$\left|\sum_{j=1}^{\infty}(\xi_{1j}^{p}+\xi_{2j}^{p})^{1/p}(\cdot\,,\varphi_j)\varphi_j\right|_s \leqslant$$
$$\leqslant\left(\left|\sum_{j=1}^{\infty}\xi_{1j}(\cdot\,,\varphi_j)\varphi_j\right|_s^p+\left|\sum_{j=1}^{\infty}\xi_{2j}(\cdot\,,\varphi_j)\varphi_j\right|_s^p\right)^{1/p}.$$

(Here ξ_{1j}^p means the real number $|\xi_{1j}|^p$ sign ξ_{1j}).

4) $|AXB|_s \leqslant \|A\|\cdot|X|_s\cdot\|B\|$, for $A,B\in B(\ell_2)$ and $X\in C$.

5) If X is a one-dimensional operator we have

$$|X|_s = \|X\| = s_1(X).$$

If instead of 4) we have the following property:

4') $|UX|_s = |XU|_s = |X|_s$ for all unitary operators U and all $X\in C$;

then $|X|_s$ is called a unitary p-norm.

Later we show that $|X|_s$ satisfies the inequality

(1) $$|X+Y|_s^p \leqslant |X|_s^p + |Y|_s^p \text{ for all } X,Y\in C.$$

Thus the name of a symmetric p-norm for $|X|_s$ in justified.

It is easy to see that a symmetric p-norm is a unitary p-norm. In fact the converse is also true for the separable symmetric p-normed spaces.

It is possible to prove by standard methods (see [4] pp. 68-69) the following result:

Proposition 1.2. a) Let $|X|_s$ be a symmetric p-norm on C. Then

$$|X|_s = |X^*|_s = |(XX^*)^{1/2}|_s = |(X^*X)^{1/2}|_s \quad \text{for all } X\in C.$$

b) If the inequalities hold

$$s_j(Y)\leqslant c\cdot s_j(X) \qquad j=1,2,3,\dots$$

where $X\in C$, Y is a compact operator and $c>0$ is a constant, then it follows that $Y\in C$ and moreover we have

$$|Y|_s \leqslant c|X|_s.$$

It is an easy consequence of Proposition 1.2 that a symmetric p-norm $|X|_s$ depends only on the singular numbers $(s_j(X))_{j=1}^{\infty}$ of the operator X.

Thus on the ideal $\mathcal{F}$ of all finite rank operators a symmetric p-norm $|X|_s$ defines a function Φ on the set of all decreasing sequences of positive numbers with at most a finite nonzero terms by the formula

$$|X|_s = \Phi(s_1(X), s_2(X), \ldots).$$

The study of this function is useful to show that $|X|_s$ verifies the inequality (1).

Let c_0 be the space of null converging sequences of real numbers and let $\hat{c}$ the subspace of c_0 consisting only of sequences with at most a finite number of nonzero terms.

Definition 1.3. A function $\Phi : \hat{c} \longrightarrow \mathbb{R}$ is called a p-normant function if the following conditions hold:

I $\Phi(\xi) > 0$ if $0 \neq \xi \in \hat{c}$.

II $\Phi(\alpha\xi) = |\alpha| \cdot \Phi(\xi)$ for $\alpha \in \mathbb{R}$ and $\xi \in \hat{c}$.

III $\Phi((\xi^p + \eta^p)^{1/p}) \leqslant (\Phi(\xi)^p + \Phi(\eta)^p)^{1/p}$ for $\xi, \eta \in \hat{c}$.

(This property is called the p-convexity of the function Φ)

IV $\Phi(1,0,0,\ldots) = 1$.

A p-normant function $\Phi(\xi)$ is called a symmetric p-normant function (briefly s.p.n.) if

V $\Phi(\xi_1, \xi_2, \ldots, \xi_n, 0, \ldots) = \Phi(|\xi_{\pi(1)}|, |\xi_{\pi(2)}|, \ldots, |\xi_{\pi(n)}|, 0, \ldots)$

for all $\xi = (\xi_i)_{i=1}^{\infty} \in \hat{c}$ and for all permutations π of the set $\{1, 2, \ldots, n\}$.

The following proposition is an easy consequence of the definition 1.3 and of the considerations made in [4] pp.71-74.

Proposition 1.4. a) If $|\xi_j| \leqslant |\eta_j|$ $j=1,2,\ldots$ hold for the vectors $\xi = (\xi_j)_{j=1}^{\infty}$, $\eta = (\eta_j)_{j=1}^{\infty}$ of $\hat{c}$, then

$$\Phi(\xi) \leqslant \Phi(\eta).$$

b) (The extension of Ky Fan's lemma). Assume that

$\xi = (\xi_j)_{j=1}^{\infty}$, $\eta = (\eta_j)_{j=1}^{\infty} \in \hat{c}$. If $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant 0$, $\eta_1 \geqslant \eta_2 \geqslant \ldots \geqslant 0$ and

$$\sum_{j=1}^{k} \xi_j^p \leqslant \sum_{j=1}^{k} \eta_j^p \qquad k = 1, 2, \ldots$$

then we have

$$\Phi(\xi) \leqslant \Phi(\eta)$$

for each s.p.n. Φ.

Easy examples of s.p.n. functions are $\Phi_\infty(\xi) = \max_{n \in \mathbb{N}} |\xi_n|$ and $\Phi_p(\xi) = \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p}$ for $\xi \in \hat{c}$. It is also clear that a s.p.n. function Φ is continuous and that

$$\Phi_\infty(\xi) \leqslant \Phi(\xi) \leqslant \Phi_p(\xi) \quad \text{for all } \xi \in \hat{c}.$$

Theorem 1.5. Let $|A|_s$ a unity p-norm on $\mathcal{F}$. Then the equality $\Phi(s(A)) = |A|_s$ for $A \in \mathcal{F}$ and $s(A) := (s_j(A))_{j=1}^{\infty}$; defines a s.p.n. function $\Phi(\xi)$. Conversely, if $\Phi(\xi)$ is a s.p.n. function, then the equality

$$|A|_{\Phi} = \Phi(s(A)) \quad \text{for } A \in \mathcal{F}$$

defines a unitary p-norm on $\mathcal{F}$.

Sketch of the proof. If $|A|_s$ is a unitary p-norm then $s_j(A) = s_j(B)$ for $j = 1,2,3,\dots$ implies that $|A|_s = |B|_s$.

Let $\Phi(\xi) = |\sum_j \xi_j^* (\cdot\, , \varphi_j)\varphi_j|_s$, where $(\varphi_j)_{j=1}^{\infty}$ is a fixed orthonormal system in ℓ_2 and $(\xi_j^*)_{j=1}^{\infty}$ is the decreasing rearrangement of a sequence $(\xi_j)_{j=1}^{\infty} \in \hat{c}$.

Then Φ is a s.p.n. function. Let's verify the property III

$$\Phi(\xi)^p + \Phi(\eta)^p = |\sum_{j=1}^{\infty} \xi_j^*(\cdot\, ,\varphi_j)\varphi_j|_s^p + |\sum_{j=1}^{\infty} \eta_j^*(\cdot\, ,\varphi_j)\varphi_j|_s^p =$$

$$= |\sum_j \xi_j(\cdot\, ,\varphi_j)\varphi_j|_s^p + |\sum_j \eta_j(\cdot\, ,\varphi_j)\varphi_j|_s^p \geqslant \text{(by the p-convexity)}$$

$$\geqslant |\sum_j (\xi_j^p + \eta_j^p)^{1/p} (\cdot\, ,\varphi_j)\varphi_j|_s^p = \Phi((\xi^p + \eta^p)^{1/p}).$$

The converse is also true using the property III for Φ . ∎

Corollary 1.6. Every unitary p-norm on the ideal $\mathcal{F}$ is a symmetric p-norm.

Now we justify that $|A|_s$ is a p-norm on $\mathcal{F}$.

Corollary 1.7. Every unitary p-norm $|A|_s$ on $\mathcal{F}$ verify the following inequality

$$|A + B|_s^p \leqslant |A|_s^p + |B|_s^p \quad \text{for } A,B \in \mathcal{F}.$$

The proof is based on the very important Theorem 2.8-[3], which asserts that

$$\sum_j s_j^p\,(A+B) \leqslant \sum_j s_j^p\,(A) + \sum_j s_j^p(B) \quad \text{for all } A,B \in \mathcal{F}.$$

Indeed

$$|A+B|_{\Phi}^p = \Phi(s(A+B))^p \leqslant \Phi((s^p(A)+s^p(B))^{1/p})^p \leqslant \Phi(s(A))^p + \Phi(s(B))^p =$$
$$= |A|_{\Phi}^p + |B|_{\Phi}^p \,. ∎$$

Definition 1.8. An ideal C of $B(\ell_2)$ endowed with a symmetric p-norm, such that C becomes a p-Banach space is called a symmetric p-normed ideal (briefly a s.p.n. ideal).

For instance each $C_p := \{X \in B(\ell_2);\ |X|_p = (\sum_{j=1}^{\infty} s_j^p(X))^{1/p} < \infty\}$ for $0 < p < \infty$ is either a symmetric p-normed space for $0 < p < 1$, or a symmetric normed space for $1 \leq p < \infty$.

We present now a general method to generate symmetric p-normed ideals.

Let Φ be a s.p.n. function and let $c_{\Phi} = \{\xi \in c_o;\ \sup_n \Phi(\xi^{(n)}) < \infty\}$ where $\xi^{(n)} = (\xi_1, \ldots, \xi_n, 0, \ldots)$ for $\xi \in c_o$.

We extend Φ to the space c_{Φ} by the formula

$$\Phi(\xi) = \lim_n \Phi(\xi^{(n)}) \quad \text{for } \xi \in c_{\Phi}.$$

<u>Definition 1.9</u>. For a s.p.n. function Φ we consider the set C_{Φ} of all compact operators X such that $s(X) = (s_j(X))_{j=1}^{\infty} \in c_{\Phi}$.

For each $X \in C_{\Phi}$ put

$$|X|_{\Phi} = \Phi(s(X)).$$

Now we can state a similar result to that of [4] p.80.

<u>Theorem 1.10</u>. <u>Let $\Phi(\xi)$ be a sp.n. function. Then the set C_{Φ} is a s.p.n. ideal with respect to the symmetric p-norm.</u>

$$|A| = |A|_C = \Phi(s(A)) \quad \underline{\text{for}} \quad A \in C_{\Phi}$$

For the ideals C_{Φ} we can extend almost all the statements proved in [4] pp.80-90.

Let's denote by C_{Φ}^o the closure of the space $\mathfrak{F}$ in C_{Φ}. Then the following theorem is true.

<u>Theorem 1.11</u>. <u>Every separable s.p.n. ideal coincides with a certain ideal C_{Φ}^o.</u>

We have already shown that a unitary p-norm on $\mathfrak{F}$ verifies the generalized triangle inequality. An important role in the proving of this fact is played by the p-convexity of the p-norm.

By Corollary 1.7 it follows, in the case p=1, that the set of properties 1)-5). Definition 1.1 is equivalent to the same set of properties, where instead of property 3) we put the usual triangle inequality.

In the case $0 < p < 1$ the situation is quite different.

The russian mathematician Y.Rotfeld shown in [6] that $C_{p,\infty} := \{T \in B(\ell_2);\ |T|_{p,\infty} = \sup_k k^{1/p} \cdot s_k(T) < \infty\}$ has an equivalent p-norm, but cannot be renormed such that it becomes a symmetric p-normed ideal. This fact shows us the importance of the property 3)of Definition 1.1.

§ 2 - Interpolation theorems for s.p.n. ideals and applications

We show some extensions of the results of J.Arazy [1], [2]. As a general rule we dont give proofs.

Let E be a separable symmetric p-normed space of sequences. Then $C_E = \{T \in B(\ell_2);\ s(T) \in E\}$ endowed with the p-norm $\|T\| = \|s(T)\|_E$, is a separable s.p.n. ideal.

We define now the triangular projection $T : C_E \longrightarrow C_E$ by the formula

$$T(A)(i,j) = \begin{cases} a(i,j) & i \leqslant j \\ 0 & \text{otherwise}, \end{cases}$$

where the matrix $(a(i,j))_{i,j=1}^{\infty}$ gives the operator $A \in C_E$ with respect to two fixed orthonormal bases $(e_n)_{n=1}^{\infty}$, $(f_n)_{n=1}^{\infty}$ in ℓ_2.

It is natural to ask about the continuity of T. We need the definition of Boyd indices for sequende spaces.

For every $m \in \mathbb{N}$, let D_m and $D_{1/m}$ be the operators defined on the symmetric p-normed space of sequences E by:

$$D_m x = (\underbrace{x(1),\dots,x(1)}_{m \text{ terms}},\ \underbrace{x(2),\dots,x(2)}_{m \text{ terms}},\dots,\underbrace{x(n),\dots,x(n)}_{m \text{ terms}},\dots)$$

$$D_{1/m} x = \Big(\sum_{i=1}^{m} x(i)/m,\ \sum_{i=m+1}^{2m} x(i)/m,\dots,\sum_{i=(n-1)m+1}^{nm} x(i)/m,\dots\Big).$$

The Boyd indices of a symmetric p-normed space E are given by

$$p_E = \sup_{m \in \mathbb{N}} \frac{\log m}{\log \|D_m\|}, \qquad q_E = \inf_{m \in \mathbb{N}} \frac{\log 1/m}{\log \|D_{1/m}\|}$$

We remark that $p_{\ell_r} = q_{\ell_r} = r$.

Let's recall that a p-Banach space E is called interpolation space for the pair (F,G) if every linear operator which is bounded on these both spaces is also bounded on the space E. As in the Corollary 3.4 - [1] we can prove the following result.

Proposition 2.1. *Let $p \leqslant p_1 < q_1 \leqslant \infty$ and let E be a symmetric p-normed space of sequences. If $p_1 < p_E$ and $q_E < q_1$ then C_E is an interpolation space for the pair (C_{p_1}, C_{q_1}).*

Now we can prove the main result.

Theorem 2.2. *Let E be a symmetric p-normed space of sequences. The triangular projection T is bounded on C_E if and only if $1 < p_E \leqslant q_E < \infty$.*

Proof. If $1 < p_E \leqslant q_E < \infty$, let p_1, q_1 such that $1 < p_1 < p_E \leqslant q_E <$

$< q_1 < \infty$. Since T is bounded on C_{p_1} and on C_{q_1} (see Proposition 4.2 -[1]) and since, by Proposition 2.1, C_E is an interpolation space for the pair (C_{p_1}, C_{q_1}), it follows that T is bounded on C_E .

Let now T be bounded on C_E and let $M = \|T\| < \infty$. We show that $1 < p_E$ (the other inequality can be proved likewise). If $p_E < 1$, by Proposition 4.2 -[1] it follows that there exists a matrix $y = (y(i,j))_{i,j=1}^{\infty}$ such that $\|y\|_{p_E} = 1$, $\|Ty\|_{p_E} \geq 4M$ and $y(i,j) \neq 0$ for a finite number of indices (i,j). Let $n \in \mathbb{N}$ be such that $y(i,j) = 0$ if $\max(i,j) > n$.

By Theorem 3.28 -[5] it follows that $\ell_{p_E}(n)$ are uniformly contained (modulo the constants $1-\varepsilon$, $1+\varepsilon$) in E, $\ell_{p_E}(n)$ being generated by n disjoint functions having the same distribution function.

Consequently there exists n normalized vectors $(x_j)_{j=1}^{n}$ of E having the same distribution, which satisfy the inequality:

$$(*) \qquad (2/3)\Big(\sum_{j=1}^{n} |a_j|^{p_E}\Big)^{1/p_E} \leq \Big\|\sum_{j=1}^{n} a_j x_j\Big\|_E \leq (4/3)\Big(\sum_{j=1}^{n} |a_j|^{p_E}\Big)^{1/p_E}$$

for all the scalars $(a_j)_{j=1}^{\infty}$.

Define now, for $1 \leq i,j \leq n$ the matrix $z_{i,j}$ which is a $n \times n$ operator-matrix and whose unique nonzero entry is the element of the coordinates (i,j) equal to x_1 (where x_1 is identified with the diagonal matrix $(x_1(i))_{i=1}^{\infty}$). Let $a = (a(i,j))_{i,j=1}^{\infty}$ a $n \times n$ matrix.

We claim that

$$(**) \qquad (4/3)\|a\|_{p_E} \geq \Big\|\sum_{i,j} a(i,j) z_{i,j}\Big\|_{C_E} \geq (2/3)\|a\|_{p_E} ,$$

where the norms are calculated in the space C_{p_E}.

Indeed, let $u = (u(i,j))_{i,j=1}^{\infty}$ and $v = (v(i,j))_{i,j=1}^{\infty}$ two unitary $n \times n$ matrices such that $b = uav = \operatorname{diag}(s_j(a))_{j=1}^{n}$.

Let $\tilde{u}$, $\tilde{v}$ the $n \times n$ operator-matrices whose (i,j)-entries are respectively $u(i,j)\cdot I$ and $v(i,j)\cdot I$. It is clear that $\tilde{u}$, $\tilde{v}$ are unitary operators and that, for $\tilde{a} = \sum_{i,j} a(i,j) z_{i,j}$, then

$$\tilde{u}\,\tilde{a}\,\tilde{v} = \operatorname{diag}(s_j(a)x_1)_{j=1}^{n} .$$

It follows that $\|\tilde{a}\|_{C_E} = \|\tilde{u}\,\tilde{a}\,\tilde{v}\|_{C_E} = \|\operatorname{diag}(s_j(a)x_1)_{j=1}^{n}\|_{C_E} =$

$= (\text{since } (x_j)_{j=1}^n \text{ have the same distribution}) = \| \sum_{j=1}^n s_j(a)x_j \|_E$,

$$(4/3)\|a\|_{p_E} = (4/3)\left(\sum_{j=1}^n (s_j(a))^{p_E}\right)^{1/p_E} \geqslant (\text{by } (*)) \geqslant \|\sum_{j=1}^n s_j(a)x_j\|_E =$$

$$= \|\sum_{i,j} a(i,j)z_{i,j}\|_{C_E} \geqslant (2/3)\left(\sum_j (s_j(a))^{p_E}\right)^{1/p_E} = (2,3)\ \|a\|_{p_E} .$$

Thus (**) is proved.

Let now $\tilde{y} = \sum_{1\leqslant i,j\leqslant n} y(i,j)z_{i,j}$. Then $\tilde{y} \in C_E$ and

$$(***) \qquad T\,\tilde{y} = \sum_{1\leqslant i\leqslant j\leqslant n} y(i,j)z_{i,j} = \widetilde{T\,y}.$$

Hence

$$M = \|T\| \geqslant \|Ty\|_{C_E} \cdot \|y\|_{C_E}^{-1} \geqslant (\text{by } (**) \text{ and } (***)) \geqslant (\tfrac{1}{2})\ \|Ty\|_{p_E}\ \|y\|_{p_E}^{-1} = 2\,M ,$$

that is we obtained a contradiction. ∎

We present now an example of a non locally-convex space of type C_E such that $1 < p_E \leqslant q_E < \infty$.

Let $0 < q < 1 < p$, $1 + 1/p > 1/q$ and let $\ell_{p,q} := \{x \in c_0;\ |x|_{p,q} = = (\sum_{n=1} x^*(n)^q \cdot n^{q/p-1})^{1/q} < \infty\}$.

Then $C_{p,q} := C_{\ell_{p,q}}$ is our space.

Indeed $\ell_{p,q}$ is a non locally-convex space, thus $C_{p,q}$ is also a non locally-convex space.

Using the elementary inequalities $k^{q/p} - (k-1)^{q/p} \leqslant k^{q/p-1}$ and $(km+1)^{q/p} - [(k-1)m+1]^{q/p} \geqslant (q/2p)\cdot k^{q/p-1}\cdot m^{q/p}$ for $k,m \geqslant 1$, we get that

$$(1/2)^{1/q}\cdot m^{1/p} \leqslant \|D_m\|_{p,q} \leqslant m^{1/p}(p/q)^{1/q} .$$

Consequently $p_{\ell_{p,q}} = p$. It is sufficient now to show that $q_{\ell_{p,q}} < \infty$. But $\|D_{1/m}\|_{p,q} \leqslant m^{1/q-1/p-1}$ for every $m \geqslant 1$, that is $\|D_{1/m}\|_{p,q} < 1$. Hence $q_{\ell_{p,q}}$.

Recall now that a topological vector space X is called a <u>primary</u> space of $X = Y \oplus Z$ implies that either $Y \approx X$ or $Z \approx X$.

Using essentially the same proof as in [2] we can state:

<u>Theorem 2.3</u>. <u>The spaces</u> C_p, <u>where</u> $0 < p < 1$, <u>are primary</u>.

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