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ON CENTERS AND STATE SPACES OF LOGICS

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Abstract. Let $C(L)$ (resp. $\mathcal{S}(L)$) denote the center (resp. the state space) of a quantum logic L . Given two quantum logics P , Q , we consider the possibility of constructing a logic L with $C(L)= C(P)$ and $\mathcal{S}(L)= \mathcal{S}(Q)$. We succeed if $\mathcal{S}(Q)$ is compact or if $C(P)$ is of special type .

1. Introduction. In the logico-algebraic approach to the foundations of quantum physics , we identify the event structure of a quantum system with a σ-orthomodular partially ordered set L (called a logic) . The set of states is then represented by the set $\mathcal{S}(L)$ of all σ-additive (probability) measures on L (see e.g. [3] , [7]) . The events of the system which are "absolutely comparable" correspond to the center $C(L)$ of L . As known, $C(L)$ is a σ-Boolean subalgebra of L .

Suppose that we look for a system with a given interplay of the center and the set of states . Expressed in the mathematical language , we ask if for given two logics P, Q there exists a logic L such that $C(L)$ is σ-Boolean izomorphic to $C(P)$ and $\mathcal{S}(L)$ is affinely homeomorphic to $\mathcal{S}(Q)$. We construct such a logic L if $C(P)$ is a σ-Boolean algebra of subsets of a set and $\mathcal{S}(Q)$ is compact (when understood as a subset of the topological linear space R^L) . If $\mathcal{S}(Q)$ is not compact we have been able to answer the question only for special types of $C(P)$.

2. Notions and results. Let us first recall basic definitions.

Definition 1 : A logic is a set L endowed with a partial ordering $\leq$ and a unary operation $'$ such that

(i) $0,1 \in L$ (L possesses a least and a greatest element),
(ii) $a \leq b \implies b' \leq a'$ for any $a,b \in L$,
(iii) $a = (a')'$ for any $a \in L$,
(iv) $a \vee a' = 1$ and $a \wedge a' = 0$ for any $a \in L$ (the symbols $\vee$, $\wedge$

mean the lattice-theoretic operations induced by $\leq$) ,
(v) $\bigvee_{i=1}^{\infty} a_i$ exists in L whenever $a_i \in L$, $a_i \leq a_j'$ for $i \neq j$,
(vi) $b = a \vee (b \wedge a')$ whenever $a,b \in L$, $a \leq b$.

For examples of logics may serve the σ-Boolean algebras or the lattice of projectors of a Hilbert space . In what follows we reserve the symbol L for logics. One can prove easily (see e.g. [3]) that if $a,b \in L$, $a \leq b'$ then $a \vee b$, $a \wedge b$ exists in L .

Definition 2 : Two elements $a,b \in L$ are called compatible if there are three elements $c,d,e \in L$ such that $c \leq d'$, $d \leq e'$, $e \leq c'$ and $a = c \vee d$, $b = c \vee e$. An element $a \in L$ is called central if a is compatible with any element of L . We denote by C(L) the set of all central elements of L and call C(L) the center of L .

Proposition 1 : The set C(L) with the operations $'$, $\vee$, $\wedge$ inherited from L is a σ-Boolean algebra .

Proof : The set C(L) is contained in any maximal σ-Boolean subalgebra of L (see [1]) . Since C(L) is obviously the intersection of all maximal σ-Boolean subalgebras of L , we obtain that C(L) is also a σ-Boolean subalgebra of L .

Definition 3 : Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics. Denote by $\prod_{\alpha \in I} L_\alpha$ the ordinary Cartesian product of the sets L_α and endow the set $\prod_{\alpha \in I} L_\alpha$ with the relation $\leq$ and the unary operation $'$ as follows. If $k = \{k_\alpha \mid \alpha \in I\} \in \prod_{\alpha \in I} L_\alpha$ and $h = \{h_\alpha \mid \alpha \in I\} \in \prod_{\alpha \in I} L_\alpha$ then $k \leq h$ (resp. $k' = h$) if and only if $k_\alpha \leq h_\alpha$ (resp. $k_\alpha' = h_\alpha$) for any $\alpha \in I$. The set $\prod_{\alpha \in I} L_\alpha$ with the above defined $\leq$, $'$ is called the product of the collection $\{L_\alpha \mid \alpha \in I\}$.

Proposition 2 : Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics . Then $\prod_{\alpha \in I} L_\alpha$ is a logic . If $C(L_\alpha) = \{0,1\}$ for any $\alpha \in I$ then $C(\prod_{\alpha \in I} L_\alpha)$ is σ-Boolean isomorphic to the σ-Boolean algebra of all subsets of I .

The proof of Proposition 2 is easy .

Definition 4 : A state on a logic L is a mapping $s : L \to \langle 0,1 \rangle$ such that (i) $s(1) = 1$, (ii) if $\{a_i \mid i \in N\}$ is a sequence of mutually orthogonal elements of L then $s(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} s(a_i)$.

Let us denote by $\mathcal{S}(L)$ the set of all states on L . Basic facts and some deeper properties of $\mathcal{S}(L)$ may be found in [2],[5] and [6] . In what follows we allow ourselves to assume that the reader is well acquainted with the results and the proof technique of the paper [6] .

Definition 5 : A logic is called poor (resp. rigid) if $\mathcal{S}(L) = \emptyset$ (resp. $|\mathcal{S}(L)| = 1$) .

It is known that there are (finite) examples of poor and rigid logics (see [2] , [6]) .

Proposition 3 : Suppose that L is a poor logic . Put $L_\alpha = L$ for any $\alpha \in I$. Then $\prod_{\alpha \in I} L_\alpha$ is also a poor logic .

Proof : Take the mapping $f: L \to \prod_{\alpha \in I} L_\alpha$ such that $f(k) = (k,k,k,\dots)$ for any $k \in L$. If $s \in \mathcal{S}(\prod_{\alpha \in I} L_\alpha)$ then $sf \in \mathcal{S}(L)$.

Definition 7 : A mapping $f: L_1 \to L_2$ is called an embedding if f injective and the following requirements are satisfied

(i) $f(1) = 1$,

(ii) $f(a') = f(a)'$ for any $a \in L$,

(iii) $a \leq b$ if and only if $f(a) \leq f(b)$,

(iv) if $a \leq b'$ then $f(a \vee b) = f(a) \vee f(b)$.

Proposition 4 : Any logic can be embedded in a poor logic with trivial center .

Proof : Let L_1 be a logic . Take a poor logic M and form the disjoint union $L_1 \cup M$. If we identify the 0 , 1 in L_1 with the 0 , 1 in M , we obtain the desired logic .

We are now ready to state our first result .

Theorem 1 : Let P , Q be logics . Let $C(P)$ be a σ-Boolean algebra of subsets of a set and let $\mathcal{S}(Q)$ be compact . Then there exists a logic L such that $C(L) = C(P)$ and $\mathcal{S}(L) = \mathcal{S}(Q)$.

Proof : Since $\mathcal{S}(Q)$ is compact , we may find a logic R such that $\mathcal{S}(R) = \mathcal{S}(Q)$, $C(R) = \{0,1\}$ and any σ-Boolean subalgebra of R is finite (see [6]) . Denote the poor extension of R by T (Proposition 4) . Write $C(P) = (A, \Sigma)$ and take a point $a \in A$. Put $L_c = T$ if $c \in A - \{a\}$, $L_a = R$. Consider the logic $V = \prod_{d \in A} L_d$. The required logic L will be a sublogic of V . We are going to describe the elements of L . An element $r \in V$ belongs to L if(and only if)there exists a countable partition $\mathcal{P}$ of A , $\mathcal{P} = \{A_i \mid i \in N\}$, such that $A_i \in B$ for any $i \in N$, and $r_p = r_q$ provided $\{p,q\} \subset A_i$ for an index $i \in N$. We must show that L is a logic with the property $C(L) = C(P) = (A, \Sigma)$ and $\mathcal{S}(L) = \mathcal{S}(R) (= \mathcal{S}(Q))$.

Let us first show that L is a logic . Evidently , $1 \in L$ and if $k \in L$ then $k' \in L$. If $k,h \in L$ and $k \geq h$ then $k = h \vee (k \wedge h')$. Indeed, if $\mathcal{P}$, $\mathcal{R}$ are partitions corresponding to k,h then $\mathcal{P} \cap \mathcal{R}$ is the (countable) partition correspnding to $k' \wedge h$. It remains to show that any sequence $\{k_i \mid i \in N\}$ of mutually orthogonal elements has the least upper bound in L . This rather technical but essentially simple part of the proof is left to the reader . (One uses the fact that any σ-Boolean subalgebra of R

is finite) .

Let us now check that $C(L) = (A, \Sigma)$. Since $C(L_d) = \{0,1\}$ for any $d \in A$, we see that any central element of L has only the elements 0 , 1 for the coordinates . One can show easily that $k = \{k_d \mid d \in A\}$, where any k_d is either 0 or 1 , belongs to L if and only if $D = \{d \mid k_d = 1\} \in \Sigma$. This implies that $C(L) = (A, \Sigma)$.

It remains to show that $\mathcal{S}(L) = \mathcal{S}(R)$. To this end, we need to exhibit an affine homeomorphism $g : \mathcal{S}(L) \rightarrow \mathcal{S}(R)$. Assume that $s \in \mathcal{S}(L)$. For any $r \in R$ we denote by k^r the element of L which has r for all its coordinates . Define $g(s)$ such that $g(s)(r) = s(k^r)$. We have to show that g is injective .

Assume that $g(s_1) = g(s_2)$. Take an element $k \in L$ and assume that $\mathcal{P}$ is the partition corresponding to k . Let A_1 be be such an element of $\mathcal{P}$ that $a \in A_1$. Denote by $h = \{h_d \mid d \in A\}$ the element of L with $h_d = 0$ if $d \in A_1$, $h_d = 1$ otherwise . It follows from Proposition 3 that $s_1(k \wedge h) = s_2(k \wedge h) = 0$. Since we have $g(s_1) = g(s_2)$, we see that $s_1(k) = s_1(k \wedge h') = s_2(k)$. Therefore the mapping g is injective and the proof is complete .

The method of the above proof , applied with complete succes in [4] for the case of finitely additive states, requires herethe assupution of compactness of $\mathcal{S}(Q)$. What may go wrong in the construction is the σ-completeness of L . The assumption on the compactness of $\mathcal{S}(Q)$ is of course very restrictive - if e.g. $\mathcal{S}(Q)$ does not have enough extreme points then $\mathcal{S}(Q)$ is not compact (Krein-Milman theorem) . We do not know if (how) one can alter the construction to obtain the theorem for general $\mathcal{S}(Q)$. What can be seen quite easily is that the method works if we restrict ourselves to certain special centers of P . Let us mention two situations .

<u>Theorem 2</u> : Let P , Q be logics . If $C(P) = \exp S$ for a set S then there is a logic L such that $C(L) = C(P)$ and $\mathcal{S}(L) = \mathcal{S}(Q)$.

The next theorem says that the countable-cocountable-type-σ-algebras may be also allowed for $C(P)$.

<u>Theorem 3</u> : Let P , Q be logics. Let $C(P)$ has the following property : If $\mathcal{P}_n = \{A_n , B_n\}$ is a sequence of two-element-partitions of $C(P)$ then there exists a countable partition of $C(P)$ which refines any $\mathcal{P}_n$. Then there exists a logic L such that $C(L) = C(P)$ and $\mathcal{S}(L) = \mathcal{S}(Q)$.

Let us observe in conclusion an amusing corollary of Theorem 1 - the existence of poor (resp. rigid) logics with arbitrary centers .

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