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RELATIONS BETWEEN FINSLER AND AFFINE CONNECTIONS

L.Tamássy - B.Kis

We consider a Finsler connection $\Gamma_F$ as a linear connection in the vertical subbundle $V\tau B$ of $\tau tl\tau B$ , $B$ denoting a differentiable manifold, $\tau B$ its tangent bundle, and $tl\tau B$ the total space of $\tau B$ . 1/ Of course such a Finsler connection can be extended in many ways to a linear connection $\Gamma$ in $\tau tl\tau B$ . V. Oproiu [3] gave a way of such an extension in which the connection in $\tau tl\tau B$ is uniquely etermined by the Finsler connection.

In this paper we investigate another way of extension using beside the Finsler connection a connection $\bar{\Gamma}$ in $\tau B$ . Moreover we investigate the inverse problem: given a linear connection $\Gamma$ in $\tau tl\tau B$ , when is this an extension in our above sense of a Finsler connection $\Gamma_F$ , and what is the connection $\bar{\Gamma}$ used at this extension. Thus we touch upon the question, when does the restriction to the vertical subspace $V\tau B$ of a linear connection $\Gamma$ in $\tau tl\tau B$ yield a Finsler connection .

In §1. we collect and partially supplement the notions and tools used in our investigations. Whitney sums and Whitney decompositions of connections have an important role throughout in our investigations. §2 deals with the extensions, and mainly with the mentioned inverse problem leading to a system of partial differential equations whose integrability is also investigated.

## §1. Connections, Whitney sums and Whitney decompositions

1. Bundles. If $\eta=(M,N,\pi)$ is a bundle, then $tl\eta \doteq M$ denotes the total space of $\eta$ , $bs\eta \doteq N$ is the base space of $\eta$, and $pr\eta \doteq \pi : M \to N$ the projection map. Manifolds, bundles and maps are supposed to be of class $C^\infty$ throughout the paper if not otherwise stated. If $\varphi$ is map of the bundle $\xi$ on the bundle

1/ There are several slightly different definitions a Finsler connection. In this paper we use the above one.

, then $\xi$ is the domain of $\varphi$, $\mathrm{dom}\,\varphi \doteq \xi$. If the diagram

$$\begin{tikzcd} tl\,\xi \arrow[r, "tl\,\varphi"] \arrow[d, "pr\,\xi"'] & tl\,\eta \arrow[d, "pr\,\eta"] \\ bs\,\xi \arrow[r, "bs\,\varphi"] & bs\,\eta \end{tikzcd}$$

is commutative $tl\,\varphi$ and $bs\,\varphi$ denoting the map showed by the figure, then $\varphi$ is called a bundle map.

$\tau M$ denotes the tangent bundle of the manifold $M$, and we also use the notations $TM \doteq tl\,\tau M$, $\tau_M \doteq pr\,\tau M$. The tangent bundle of a bundle $\xi$ is $\tau\xi \doteq (T\,tl\,\xi, T\,pr\,\xi, T\,bs\,\xi)$ where $T\,pr\,\xi$ denotes the differential of the map $pr\,\xi$ denoted sometimes also by $d\,pr\,\xi$. $V\xi \doteq (\mathrm{Ker}\,pr\,\tau\xi, \tau_{tl\,\xi}, tl\,\xi)$ is called the vertical subbundle of $\tau\xi$.

Let $\xi$ be a bundle, $M$ a differentiable manifold, and $\Psi : M \to bs\,\xi$ a map. If the elements $\alpha_1, \alpha_2$ of the set $\square(\xi, \Psi) \doteq \{\alpha : \eta \to \xi \mid \alpha$ is a bundle map, $bs\,\alpha = \Psi\}$ satisfy the property that there exists a bundle map $\sigma : \mathrm{dom}\,\alpha_1 \to \mathrm{dom}\,\alpha_2$ for which $\alpha_1 = \alpha_2 \circ \sigma$ then we say that $\alpha_2$ factorizes $\alpha_1$.

$\sigma$ will be denoted by $\mathrm{Fact}_\xi(\alpha_1, \alpha_2)$. The unique element of the set $\square(\xi, \Psi)$ factorizing every element of $\square(\xi, \Psi)$ is denoted by $ad_\xi \Psi$, and $\mathrm{Fact}_\xi(\alpha, ad_\xi \Psi)$ by $\mathrm{Fact}_\xi\,\alpha$. $\mathrm{dom}\,ad_\xi \Psi$ is called the pull-back bundle of $\xi$ by $\Psi$ and we denote it by $\Psi^!\xi$. It is well known that $V\xi$ and $(pr\,\xi)^!\xi$ are canonically isomorphic. We denote it by

(1) $$j_\xi : (pr\,\xi)^!\xi \to V\xi$$

A vector bundle $\xi$ is the Whitney sum of the vector bundles $\xi^1, \xi^2$; $bs\,\xi^1 = bs\,\xi^2$ if there exist bundle homomorphisms $p^\nu : \xi \to \xi^\nu$, $\iota^\nu : \xi \to \xi^\nu$ $(\nu = 1,2)$ satisfying the conditions:

(2) $$p^\nu \circ \iota^\mu = \begin{cases} \emptyset & \text{if } \nu \neq \mu \\ id_\xi & \text{if } \nu = \mu \end{cases} \quad \text{and} \quad \iota^1 \circ p^1 + \iota^2 \circ p^2 = id_\xi$$

As it is well known, the sequences $0 \to \xi^1 \xrightarrow[\iota^1]{} \xi \xrightarrow[p^2]{} \xi^2 \to 0$ and $0 \leftarrow \xi^1 \xleftarrow{p^1} \xi \xleftarrow{\iota^2} \xi^2 \leftarrow 0$ are short exact sequences, and the corresponding maps split each other. We indicate this fact by the diagram

(3) $$0 \longleftrightarrow \xi^1 \underset{\iota^1}{\overset{p^1}{\longleftrightarrow}} \xi \underset{p^2}{\overset{\iota^2}{\longleftrightarrow}} \xi^2 \longleftrightarrow 0$$

(3) obviously determines a Whitney sum, and we will give Whitney sums mostly in this form, and say that (3) is a dual short exact sequence. Let $\xi^1$ and $\xi^2$ be subbundles of $\xi$ such that every fibre of $\xi$ over an $x \in bs\,\xi$ is the direct sum of the appropriate

fibres of $\xi^1$ and $\xi^2$. Furthermore let $p^\nu : \xi \to \xi^\nu$ and be the natural projections and inclusions. Then (3) is a dual short exact sequence. In this case we write $\xi = \xi^1 \oplus \xi^2$. This is the classical Whitney sum of $\xi^1$ and $\xi^2$. - In the subsequent part of the paper bundles are always vector bundles.

2. Connections. A connection $\Gamma$ on a vector bundle $\xi$ is the dual short exact sequence

(4) $$0 \longleftrightarrow (pr\,\xi)^! \xi \underset{\dot{j}_\xi}{\overset{v_\xi}{\longleftrightarrow}} \tau t \xi \underset{p_\xi}{\overset{h_\xi}{\longleftrightarrow}} (pr\,\xi)^! \tau b \xi \longleftrightarrow 0$$

where $\dot{j}_\xi$ is the inclusion map induced by $j_\xi$, and $p_\xi \doteq \mathrm{Fact}_{\tau b \xi}\, d(pr\,\xi)$. The Dombrovski map of the connection $\Gamma$ is

(5) $$K_\nabla \doteq ad_\xi\, pr\,\xi \circ v_\xi : \tau t \xi \to \xi$$

The covariant derivative $\nabla$ associated to $\Gamma$ is

(6) $$\nabla(v, \sigma) \doteq (K_\nabla \circ T\sigma)$$

A connection is uniquely determined by its Dombrovski map.

Proposition 1. The bundle homomorphism $K : \tau t \xi \to \xi$ is the Dombrovski map of some connection $\Gamma$ iff

(7) $$K_{|V\xi} = ad_\xi\, pr_\xi \circ r_\xi$$

where $r_\xi = \dot{j}_\xi^{-1}$.

Proof. Suppose that $K$ is the Dombrovski map of a connection (4). Then $K_{|V\xi} = (ad_\xi\, pr\,\xi \circ v_\xi)_{|V\xi} = ad_\xi\, pr_\xi \circ (v_\xi)_{|V\xi} = ad_\xi\, pr\,\xi \circ r_\xi$ because of $v_\xi \circ \dot{j}_\xi = id_{(pr\,\xi)^! \xi}$, and $\mathrm{Im}\, \dot{j}_\xi = V\xi$.

Conversely, suppose that the bundle homomorphism $K : \tau t \xi \to \xi$ satisfies the condition (7). Now $\mathrm{Ker}\, K = \mathrm{Ker}\, v_\xi$ because $ad_\xi\, pr\,\xi$ is a bijection on the fibres. Moreover $K(V\xi) = (ad_\xi\, pr\,\xi \circ r_\xi)(V\xi) = (ad_\xi\, pr\,\xi)((pr\,\xi)^! \xi) = \xi$, thus $V\xi \oplus \mathrm{Ker}\, v_\xi = \tau t \xi$ therefore we can write every $z \in \tau t \xi$ in the form $z = x + y$, where $x \in V\xi$ and $y \in \mathrm{Ker}\, v_\xi = \mathrm{Ker}\, K$. Now, $v_\xi(z) = v_\xi(x) + 0 = (v_{\xi|V\xi})(x) = r_\xi(x)$, which has already been given. Thus $v_\xi(z)$ is given if we know $\mathrm{Ker}\, K$. On the other hand $v_\xi$ uniquely determines the connection (4).

Let $\xi^1$ be a vector-subbundle of $\xi$. We say that <u>the connection $\Gamma$ is invariant on the subbundle $\xi^1$</u>, if

$\nabla(v, \sigma) \in \mathrm{sec}\, \xi^1$ for every $v \in \mathrm{sec}\, \tau b \xi$ and for every $\sigma \in \mathrm{sec}\, \xi^1$,

i.e. if $(K_\nabla \circ T\sigma)(v) \in \xi^1$. This is equivalent with the property

$$(K_\nabla \circ z)(v) \in t\xi^1 \text{ for every } v \in \tau b \xi,\ z \in t \tau \xi^1.$$

Let $\alpha : \xi \to \xi'$ be an isomorphism between the bundles $\xi$ and $\xi'$. Now a connection $\Gamma$ on $\xi$ induces a connection $\Gamma'$ on $\xi'$ by the Dombrovski map $K_{\nabla'} \doteq \alpha \circ K_{\nabla} \circ (d\alpha^{-1})$

We denote $\operatorname{Im} h_\xi$ by $H\xi$ . $H\xi$ uniquely determines the connection (4). The map $p_\xi$ is a surjection /on the fibres/, and its kernel consists of the fibres of $V\xi$ . Thus $p_\xi$ is a bijection on $H\xi$ , and there exists the inverse of $p_{\xi|H\xi}$ . Denote this inverse of $p_{\xi|H\xi}$ by $h_\xi$ . The map $\dot{j}_\xi$ is an injection /on the fibres/ and its image is $V\xi$ . We know that $\dot{j}_\xi : (pr\,\xi)^! \xi \to V\xi$ is a canonical isomorphism, and the inverse of $\dot{j}_\xi$ is denoted by $v_\xi$ .

3. <u>Whitney sums and Whitney decompositions of a connection</u>. Let $\xi$, $\xi^\nu$ $(\nu = 1,2)$ be vector bundles, satisfying (3), thus $\xi$ the Whitney sum of $\xi^\nu$ . Let a connection $\Gamma^\nu$ be given on $\xi^\nu$ :

$$(8) \qquad 0 \longleftrightarrow (pr\,\xi^\nu)^! \xi^\nu \underset{\dot{j}_{\xi^\nu}}{\overset{v_{\xi^\nu}}{\rightleftarrows}} \tau\, tot\, \xi^\nu \underset{p_{\xi^\nu}}{\overset{h_{\xi^\nu}}{\rightleftarrows}} (pr\,\xi^\nu)^! \tau\, bs\, \xi^\nu \longrightarrow 0$$

and let $K_{\nabla^\nu}$ be the Dombrovski map of $\Gamma^\nu$ . Then we can give a connection $\Gamma$ on the vector bundle $\xi$ with the aid of $\Gamma^\nu$ and (3). The simplest way of giving this connection is using its Dombrovski map, defined by

$$(9) \qquad K_\nabla = \iota^1 \circ K_{\nabla^1} \circ dp^1 + \iota^2 \circ K_{\nabla^2} \circ dp^2$$

With the help of Proposition 1. we can easily see that $K_\nabla$ is in fact a Dombrovski map of some connection $\Gamma$ on $\xi$ . Moreover, if $\Gamma^1$ and $\Gamma^2$ are linear then $\Gamma$ too is so. This connection is the Whitney sum of $\Gamma^1$ and $\Gamma^2$ by (3). Conversely, let $\Gamma$ be a connection on the above vector bundle $\xi$ . Then two new connections $\Gamma^1$ and $\Gamma^2$ can be given on the vector bundles $\xi^1$ and $\xi^2$ with the aid of (3). These connections are given by their Dombrovski maps $K_{\nabla^1}$ and $K_{\nabla^2}$ :

$$(10) \qquad K_{\nabla^\nu} = p^\nu \circ K_\nabla \circ d\iota^\nu$$

$\Gamma^\nu$ are called the Whitney decomposition of the connection $\Gamma$ by (3) on the vector bundle $\xi^\nu$ . If $\Gamma$ is linear, then $\Gamma^1$ and $\Gamma^2$ are linear, too.

We say that <u>a connection $\Gamma$ on $\xi$ is invariant by (3)</u> if the Whitney sum by (3) of the Whitney decompositions of $\Gamma$ by (3) is $\Gamma$ again.

<u>Proposition</u> 2. A connection $\Gamma$ on the Whitney sum $\xi$ of $\xi^1$ and $\xi^2$ according to (3) is invariant by (3) iff $\Gamma$ is invariant on the bundles $\operatorname{Im} \iota^\nu$ $(\nu = 1,2)$.

Proof. For the necessity we get the following equation according to the formulas (9) and (10):

$$K_\nabla = (\iota^1\circ p^1)\circ K_\nabla\circ d(\iota^1\circ p^1) + (\iota^2\circ p^2)\circ K_\nabla\circ d(\iota^2\circ p^2) \tag{11}$$

Multiplying by $p^1$ and arranging our formulas we get the following equation:

$$p^1\circ K_\nabla - p^1\circ K_\nabla\circ d(\iota^1\circ p^1) = p^1\circ K_\nabla\, d(id_\xi - \iota^1\circ p^1) = p^1\circ K_\nabla\circ d(\iota^2\circ p^2) = 0$$

Thus for all $v\in \tau b_s \xi$ and for all $\sigma\in \mathrm{Acc}\,\mathrm{Im}\,\iota^2$ $\quad K_\nabla\circ d\sigma\in \mathrm{Ker}\,p^1 = \mathrm{Im}\,\iota^2$, and so $\Gamma$ is invariant on $\mathrm{Im}\,\iota^2$. Similarly $\Gamma$ is invariant on $\mathrm{Im}\,\iota^1$, too.

The sufficiency means that

$$((\iota^1\circ p^1\circ K_\nabla\circ d\iota^1\circ dp^1 + \iota^2\circ p^2\circ K_\nabla\circ d\iota^2\circ dp^2)\circ d\sigma)(v)\in \xi^v$$

for $\sigma\in \mathrm{Acc}\,\mathrm{Im}\,\iota^v$ if $(K_\nabla\circ d\sigma)(v)\in\xi^v$. But this is trivial.

§2. Finsler connections and vertical invariant connections

1. A linear connection in the vector bundle $(pr\tau B)^!\tau B$ or in $V\tau B$ is called a Finsler connection over the manifold $B$, or on the bundle $\tau B$. Let a connection $\bar\Gamma$

$$0 \longleftrightarrow (pr\tau B)^!\tau B \underset{j_{\tau B}}{\overset{v_{\tau B}}{\rightleftarrows}} \tau t\ell\tau B \underset{\bar h_{\tau B}}{\overset{h_{\tau B}}{\rightleftarrows}} (pr\tau B)^!\tau B \longleftrightarrow 0 \tag{12}$$

be given in the vector bundle $\tau B$. Since $\tau B$ is a vector bundle $\xi$ we can apply the results of §1. So according to section 1.1 and formula (1), $j_{\tau B}$ performs an isomorphism $(pr\tau B)^!\tau B \longrightarrow V\tau B$ and given a Finsler connection $\Gamma_F$ in $(pr\tau B)^!\tau B$ $\quad j_{\tau B}$ induces a connection $\overset{1}{\Gamma}$ in $V\tau B$ according to section 1.2. Similarly $h_{\tau B}$ transplants the connection $\Gamma_F$ in $(pr\tau B)^!\tau B$ into a connection $\overset{2}{\Gamma}$ in $H\tau B$ /see section 1.2./. Then we can form the Whitney sum $\Gamma$ of $\Gamma_F$ with itself by (12). This yields the same as the classical Whitney sum of $\overset{1}{\Gamma}$ ~~and~~ $\overset{2}{\Gamma}$, in $\tau t\ell\tau B$ for $\tau t\ell\tau B = V\tau B\oplus H\tau B$. This $\Gamma$ is induced by $\Gamma_F$ and $\bar\Gamma$, and is called the extension of the Finsler connection $\overset{1}{\Gamma}$ with the aid of $\bar\Gamma$. The connection $\Gamma$ determined in this way is obiously invariant on $V\tau B$ and $H\tau B$.

Now we wish to investigate the converse problem: Is every vertical invariant linear connection $\Gamma$ in $\tau t\ell\tau B$ an extension of a Finsler connection $\Gamma_F$ with the aid of an appropriate $\bar\Gamma$ ? The answer is negative. We will determine the conditions for the positive answer, and this will show that in cases when these conditions are not fulfiled, $\Gamma$ is no extension of $\Gamma_F$ by a $\bar\Gamma$.

2. A linear connection in $\tau H \tau B$ is called a vertical invariant connection, if it is invariant on the subbundle $V\tau B$. It is clear that every Finsler connection in $V\tau B$ can be extended in many ways to a vertical invariant connection in $\tau H \tau B$. The simplest example of such an extension was given in the previous paragraph.

From this point on we will compute locally. Let $(x^i, y^i)_{i,j=\overline{1,n}}$ be the natural local coordinate system in $\tau B$, determined by a local coordinate system $(x^i)$ of $B$. Then $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ is at $(x,y)$ a local base of the fibre of $\tau H \tau B$, and $(\frac{\partial}{\partial y^i})$ is the canonical base of $V\tau B$ at the same point. Be $e_i = r_{\tau B} \frac{\partial}{\partial y^i}$ then $(e_i)$ is a base in the fibres of the bundle $(pr\,\tau B)^! \tau B$.

The covariant derivative, associated to $\Gamma_F$ is

$$(13)\qquad \begin{aligned} \nabla_F \frac{\partial}{\partial x^i} &= G^k_{\alpha i}\, dz^\alpha \frac{\partial}{\partial x^k} + H^k_{\alpha i}\, dz^\alpha \frac{\partial}{\partial y^k} \\ \nabla_F \frac{\partial}{\partial y^i} &= C^k_{\alpha i}\, dz^\alpha \frac{\partial}{\partial y^k} \end{aligned} \qquad \alpha = \overline{1,2n}$$

since $\Gamma_F$ is invariant on $V\tau B$. Here $z^\alpha = x^\alpha$ if $1 \le \alpha \le n$ and $z^\alpha = y^{\alpha - n}$ if $n+1 \le \alpha \le 2n$. Thus Greek indices run from $1$ to $2n$, Latin indices run from $1$ to $n$, and the summation convention is also applied.

Now we are going to express in the local bases $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ the covariant derivatives $\overset{*}{\nabla} \frac{\partial}{\partial x^i}$ and $\overset{*}{\nabla} \frac{\partial}{\partial y^i}$ associated to a connection $\overset{*}{\Gamma}$ which is an extension of a Finsler connection $\Gamma^1$ in $V\tau B$ with the aid of an arbitrary $\bar{\Gamma}$, $\Gamma^1$ being the transplant by $\dot{j}_{\tau B}$ of an arbitrary Finsler connection $\Gamma_F$ in $(pr\,\tau B)^! \tau B$ : $K_{\dot{j}} = \dot{j}_{\tau B} \circ K_{\nabla_F} \circ (d\dot{j}^{-1}_{\tau B})$ /see section 1.2/. Thus $\overset{*}{\Gamma}$ is the Whitney sum of $\Gamma_F$ with itself by (12). It is clear that this construction yields all those vertical invariant connections which are extensions in the above sense. Then a comparison of $\overset{*}{\nabla} \frac{\partial}{\partial x^i}$ and $\overset{*}{\nabla} \frac{\partial}{\partial y^i}$ with (13) yields the sought for our conditions. First we study the maps of the dual short exact sequences (12). By the definitions /page 3 and (1) /

$$(15)\qquad \dot{j}_{\tau B}(e_i) = \frac{\partial}{\partial y^i} \qquad (i = \overline{1,n})$$

Similarly we have

$$(16)\qquad \begin{aligned} P_{\tau B}\left(\frac{\partial}{\partial x^i}\right) &= e_i \\ P_{\tau B}\left(\frac{\partial}{\partial y^i}\right) &= 0 \end{aligned}$$

since the kernel of $P_{\tau B}$ consists of the fibers of $V\tau B$ ./page 4/.

Assume now that $h_{\tau B}(e_i) = S_i^r \frac{\partial}{\partial x^r} - A_i^r \frac{\partial}{\partial y^r}$ with some functions $S_i^r$ and $A_i^r$. We know that $p_{\tau B} \circ h_{\tau B} = id_{(p \tau B)^! \tau B}$ so

$$e_i = (p_{\tau B} \circ h_{\tau B})(e_i) = p_{\tau B}(S_i^r \frac{\partial}{\partial x^r} - A_i^r \frac{\partial}{\partial y^r}) = S_i^r e_r .$$

Therefore $S_i^r \equiv \delta_i^r$. Thus

$$(17) \qquad h_{\tau B}(e_i) = g_j \doteq \frac{\partial}{\partial x^i} - A_i^s \frac{\partial}{\partial y^s}$$

Now, as we know $\mathring{j}_{\tau B} \circ v_{\tau B} = id_{\tau \mathcal{U} \tau B} - h_{\tau B} \circ p_{\tau B}$, and

$$\mathrm{Im}(id_{\tau \mathcal{U} \tau B} - h_{\tau B} \circ p_{\tau B}) = \mathrm{Im}(\mathring{j}_{\tau B} \circ v_{\tau B}) = V\tau B$$

As we have seen, $\mathring{j}^{-1}_{\tau B | V\tau B} = j^{-1}_{\tau B} = \tau_{\tau B}$ and so

$$v_{\tau B} = j^{-1}_{\tau B} \circ (id_{\tau \mathcal{U} \tau B} - h_{\tau B} \circ p_{\tau B}) = \tau_{\tau B} \circ (id_{\tau \mathcal{U} \tau B} - h_{\tau B} \circ p_{\tau B}).$$

Finally, in wiew of (16) and (17) we obtain

$$v_{\tau B}(\frac{\partial}{\partial x^i}) = \tau_{\tau B} \circ (id_{\tau \mathcal{U} \tau B} - h_{\tau B} \circ p_{\tau B})(\frac{\partial}{\partial x^i}) = A_i^s e_s$$

$$v_{\tau B}(\frac{\partial}{\partial y^i}) = e_i$$

Let the covariant derivative $\nabla_F$ associated to $\Gamma_F$ be locally:

$$(18) \qquad \nabla_F(e_i) = \Gamma_{\alpha i}^k dz^\alpha e_k$$

Now, according to (6), our constructions of $\overset{*}{\Gamma}$, and (9) we obtain

$$\overset{*}{\nabla} \frac{\partial}{\partial x^i} = K_{\overset{*}{\nabla}} \circ d(\frac{\partial}{\partial x^i}) = (\mathring{j}_{\tau B} \circ K_{\nabla_F} \circ d v_{\tau B} + h_{\tau B} \circ K_{\nabla_F} \circ d p_{\tau B}) d(\frac{\partial}{\partial x^i}) =$$

$$(\mathring{j}_{\tau B} \circ \nabla_F \circ v_{\tau B} + h_{\tau B} \circ \nabla_F \circ p_{\tau B})(\frac{\partial}{\partial x^i})$$

Using (16), (17), (18) and (15) we get

$$\overset{*}{\nabla}(\frac{\partial}{\partial x^i}) = (\mathring{j}_{\tau B} \circ \nabla_F)(A_i^s e_s) + (h_{\tau B} \circ \nabla_F)(e_i) = \mathring{j}_{\tau B}(A_i^s \Gamma_{\alpha s}^t dz^\alpha e_t + \frac{\partial A_i^t}{\partial z^\alpha} dz^\alpha e_t) +$$

(19)

$$h_{\tau B}(\Gamma_{\alpha i}^r dz^\alpha e_r) = \Gamma_{\alpha i}^r dz^\alpha \frac{\partial}{\partial x^r} + (\frac{\partial A_i^t}{\partial z^\alpha} + A_i^s \Gamma_{\alpha s}^t - \Gamma_{\alpha i}^r A_r^t) dz^\alpha \frac{\partial}{\partial y^t}$$

Similarly we have

$$(20) \qquad \overset{*}{\nabla} \frac{\partial}{\partial y^i} = (\mathring{j}_{\tau B} \circ \nabla_F \circ v_{\tau B} + h_{\tau B} \circ \nabla_F \circ p_{\tau B}) \frac{\partial}{\partial y^i} = \Gamma_{\alpha j}^r dz^\alpha \frac{\partial}{\partial y^r}$$

Thus $\overset{*}{\nabla}$ is determined by $\Gamma_{\alpha i}^r$ and $A_i^s$.

If $\Gamma$ is an extension in the considered sense, then $\Gamma$ and $\overset{*}{\Gamma}$ must be equal for certain $\Gamma_{\alpha i}^r$ and $A_i^s$. Thus from (13), (19) and (20) we obtain

$$G^{\tau}_{\alpha i} = \Gamma^{\tau}_{\alpha i}$$
$$H^{t}_{\alpha i} = \frac{\partial A^{t}_{i}}{\partial z^{\alpha}} + A^{s}_{i}\Gamma^{t}_{\alpha s} - \Gamma^{\tau}_{\alpha i} A^{t}_{\tau}$$
$$C^{k}_{\alpha i} = \Gamma^{k}_{\alpha i}$$

Denoting in parenthesis the quantities locally determining a connection we get the

Theorem: A vertical invariant connection $\Gamma(G^{\tau}_{\alpha i}, H^{\tau}_{\alpha i}, C^{\tau}_{\alpha i})$ is locally the extension of a Finsler connection $\Gamma_F(\Gamma^{\tau}_{\alpha i})$ with the aid of a connection $\bar{\Gamma}(A^{s}_{j})$ iff $G^{\tau}_{\alpha i}$ and $C^{\tau}_{\alpha i}$ coincide; $\Gamma^{\tau}_{\alpha i} = C^{\tau}_{\alpha i}$ and the partial differential equation system

$$H^{t}_{\alpha i} = \frac{\partial A^{t}_{i}}{\partial z^{\alpha}} + A^{s}_{i} C^{t}_{\alpha s} - C^{\tau}_{\alpha i} A^{t}_{\tau} \tag{21}$$

is integrable.

We note that if a connection $\Gamma$ is an extension of a $\Gamma_F$ with the aid of $\bar{\Gamma}$, this fact is independent of the local coordinate system used. Thus, if (21) is integrable in a set of local coordinate systems covering $\mathcal{T}B$, then also globally is an extension, but the differentiability of $\bar{\Gamma}$ is not yet answered. However if $\bar{\Gamma}$ /i.e. the solution of (21) / is unique, then $\Gamma$ is a global extension in our sense.

## 3. The integrability of (21).

In a linear connection on a manifold $M$

$$\nabla\left(A^{\alpha}_{\beta}\, dz^{\beta} \frac{\partial}{\partial y^{\alpha}}\right) = \left(\nabla_{\sigma} A^{\alpha}_{\beta}\right) dz^{\beta}\, dz^{\sigma} \frac{\partial}{\partial y^{\alpha}}$$

for any tensor field $A^{\alpha}_{\beta}$. It is well known / (4) pp. 124-127/ that

$$2\nabla_{[\mu}\nabla_{\rho]} A^{\alpha}_{\beta} = R_{\mu\rho\sigma}{}^{\alpha} A^{\sigma}_{\beta} - R_{\mu\rho\beta}{}^{\sigma} A^{\alpha}_{\sigma} - 2 S_{\mu\rho}^{\sigma} \nabla_{\sigma} A^{\alpha}_{\beta} \tag{22}$$

where $R$ and $S$ are the curvature an torsion tensor of the connection.

Let us define the following connection:

$$\nabla^{\circ}_{\rho} \frac{\partial}{\partial x^{i}} = 0 \quad , \quad \nabla^{\circ}_{\rho} \frac{\partial}{\partial y^{i}} = \nabla_{\rho} \frac{\partial}{\partial y^{i}} \tag{23}$$

Finally let $B^{\alpha}_{\beta}$ be given by the definition

$$B^{\alpha}_{\beta} \doteq \begin{cases} A^{\alpha}_{\beta}, \text{ if } & \alpha \le n \text{ and } \beta \le n \\ 0, \text{ in other cases.} \end{cases} \tag{24}$$

Now in view of (24), (23) and (13) the studied partial equation system (21) can be written in the form

$$(\nabla^{\circ}_{F})_{\alpha}\, B^{\tau}_{\beta} = H^{\tau}_{\alpha\beta} \tag{25}$$

where $H^{t}_{\alpha i}$ are the coefficients appearing in our theorem, while the other coefficients of $H^{\tau}_{\alpha\beta}$ are zeros.
The integrability condition of (25) is

$$\partial_{[\delta} (\nabla^{\circ}_{F})_{\alpha]} B^{\tau}_{\beta} = \partial_{[\delta} H^{\tau}_{\alpha]\beta}$$

This is equivalent with the condition

$$(\nabla^{\circ}_{F})_{[\delta} (\nabla^{\circ}_{F})_{\alpha]} B^{\tau}_{\beta} = (\nabla^{\circ}_{F})_{[\delta} H^{\tau}_{\alpha]\beta}$$

So, by (22) we get the condition

$$R_{\delta\alpha\mu}{}^{\tau} B^{\mu}_{\beta} - R_{\delta\alpha\beta}{}^{\mu} B^{\tau}_{\mu} = 2 (\nabla^{\circ}_{F})_{[\delta} H^{\tau}_{\alpha]\beta} + 2 S^{\mu}_{\delta\alpha} H^{\tau}_{\mu\beta} \tag{26}$$

for the integrability of (21). (26) is an ordinary equation system at every $z^{\alpha}$ for the unknowns $B^{\mu}_{\beta}$. If (26) has a solution for $B^{\mu}_{\beta}$ satisfying (24), then our system (21) is also integrable.

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