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SOME COMMENTS ON A CUBIC ALGEBRA

E.F. Corrigan

1. Introduction

A talk late in the afternoon might be a good time to set aside the serious topics of the earlier lectures and consider something lighter and possibly entertaining. The motivation for the problem I would like to discuss will be mentioned towards the end of the talk rather than the beginning, so as to avoid the risk of boring at least half the audience too soon. The problem is easy to state: find all possible sets of D $N \times N$ hermitian and traceless matrices, α_i $i = 1, \ldots D$, satisfying the set of equations

$$\sum_{i=1}^{D} [\alpha_i, [\alpha_i, \alpha_j]] = \sigma\alpha_j \tag{1.1}$$

for $j = 1 \ldots D$, where σ is a conveniently chosen constant. (Obviously, we could if we wished scale σ to one, but that may not be very convenient for some purposes.)

There is one solution to eq(1.1), that does not take long to find, provided D is the dimension of the Lie algebra corresponding to some Lie group G. In that case, the α's represent the Lie algebra and σ may as well be taken to be the value of the quadratic Casimir operator in the adjoint representation of the algebra. If $D = 3$, the case that we were originally motivated to study (CORRIGAN, WAINWRIGHT and WILSON, 1985), the α's represent SO(3) and $\sigma = 2$ is the natural choice for σ. However, there are other solutions too and our aim is to enumerate and classify them as fully as we can.

For any choice of D we can also check that the 'gamma' matrices satisfying

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad i, j = 1, \ldots, D \tag{1.2}$$

also satisfy the cubic equations (1.1). To see this, a convenient way is to realise (SUDBERY) that the double commutator can be rewritten as the difference of two double anticommutators:

$$[\alpha_i, [\alpha_i, \alpha_j]] = \{\{\alpha_i, \alpha_i\} \alpha_j\} - \{\{\alpha_i, \alpha_j\}, \alpha_i\}.$$

Hence, using eq(1.2) we get

$$\sum_i [\alpha_i, [\alpha_i, \alpha_j]] = 4(D-1)\alpha_j,$$

as desired, choosing $\sigma = 4(D-1)$ to agree with (1.1).

It is also interesting to note that the octonions, which cannot of course be represented by matrices, yield a solution to eq(1.1) for $D = 7$. Because of the sum their non associativity fails to prevent the left hand side collapsing down to reproduce α_j.

Besides these relatively straightforward solutions there are yet more which can be described in detail for $D = 3$, $N = 2,3$. However, a discussion of these is perhaps best postponed till after making some further more general comments about eq(1.1).

2. General remarks

Our set of equations has a large symmetry. In fact an $SO(D) \times SU(N)$ symmetry where the $SO(D)$ acts on the explicit labels i,j and the $SU(N)$ conjugates each of the α's. In other words, given one solution α_i another continuously connected to it is given by

$$\alpha_i' = O_{ij}\, g^+ \alpha_j\, g, \tag{2.1}$$

with $O \in SO(D)$ and $g \in SU(N)$.

To distinguish solutions we shall need some invariants. For example, the quantity $M_\alpha = \sum_i \gamma^i \otimes \alpha^i$ has eigenvalues invariant under (2.1) and allows us to distinguish inequivalent sets of solutions provided the corresponding eigenvalues of M differ. However, it can happen that M_α and $M_{\alpha'}$ have the same eigenvalues but are nevertheless inequivalent under the restricted group transformation (2.1).

In three dimensions ($D = 3$) it turns out that the two invariants

$$q = \mathrm{tr}\, \underline{\alpha}^2 , \qquad t = -\mathrm{tr}(\underline{\alpha} \cdot \underline{\alpha} \wedge \underline{\alpha}) , \tag{2.2}$$

appear to be enough to distinguish all the solutions we have found to date. Also, we note for any solution to eq(1.1), with $\sigma = 2$,

$$|t| \leqslant q \tag{2.3}$$

with equality, $t = \pm q$, if and only if $\pm\underline{\alpha}$ represents the Lie algebra of $SU(2)$. To see the relation (2.3), consider

$$0 \leq \mathrm{tr}\,\{(\underline{\alpha} \pm i\,\underline{\alpha} \wedge \underline{\alpha}) \cdot (\underline{\alpha} \pm i\,\underline{\alpha} \wedge \underline{\alpha})$$

$$= \mathrm{tr}(\underline{\alpha}^2) \pm 2\,i\mathrm{tr}\,(\underline{\alpha} \cdot \underline{\alpha} \wedge \underline{\alpha}) - \mathrm{tr}\,\{\,\underline{\alpha} \wedge \underline{\alpha} \cdot \underline{\alpha} \wedge \underline{\alpha}\,\}$$

$$= 2q \pm 2t,$$

clearly the right hand side vanishes if and only if

$$i\underline{\alpha} = \underline{\alpha} \wedge \underline{\alpha},$$

as required.

Another way that we might think of distinguishing solutions is by their behaviour under what we might call the 'commutator' mapping. Suppose we have a solution to eq(1.1) and construct a new set (of at most $\frac{D(D-1)}{2}$ distinct) matrices by forming all the commutators of the α's with each other. Then, it sometimes happens that the new set also satisfies (a suitably enlarged) set of cubic equations. This is obviously true if the α's are arranged so that $\pm\alpha$ represents a lie algebra, but is also true in other circumstances too. For example, the γ-matrices (1.2) lead to the set of commutators $i\,[\alpha^i, \alpha^j]$, which represent so(D) and thus satisfy an enlarged set of equations as a Lie algebra. The γ-matrices also provide more intricate examples.

Consider the 'ring' of γ matrix products

$$(\alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_D) = i(\gamma^1\gamma^2, \gamma^2\gamma^3, \gamma^3\gamma^4, \ldots, \gamma^D\gamma^1). \tag{2.4}$$

This collection of products also provides a nice solution to eqs(1.1), with $\sigma = 8$.(KENT). Their behaviour under the commutator mapping depends somewhat on D and needs a little work to check it. Before doing so it should be pointed out that solutions which differ by a change of sign of an even number of the α's are equivalent to the α's themselves under (2.1), whilst an odd number of changes of sign leads to an inequivalent solution. However, in considering commutators there is no way of canonically fixing signs so we shall enlarge (2.1) to include the disconnected pieces obtained by swapping an odd number of signs - effectively replacing SO(D) by O(D).

For the ring (2.4) evaluating the commutators produces another ring of matrix products, this time with the products containing factors spaced by two. That is we obtain

$$(\alpha_1', \ldots, \alpha_D') = i(\gamma^1\gamma^3, \gamma^3\gamma^5, \ldots, \gamma^D\gamma^2, \gamma^2\gamma^4, \ldots \gamma^{D-1}\gamma^1), \tag{2.5}$$

clearly another solution. However, only if D is odd the new ring (2.5) is a single ring, for D even it is actually a pair of rings of length $D/2$. Moreover, for D odd a repeated application of the commutator mapping produces the original solution again after k steps, where $2^k = -1 \mod D$. It is also the case that at each step the new ring is equivalent to the original one in the sense of (2.1). This is because the γ matrices can be permuted amongst themselves using a transformation in SU(N), (N = dim γ). Thus, for D odd all the solutions lie on the same orbit. For D even this is not so. We have already remarked that the original ring breaks up into a pair of rings of equal length. If the length of these subrings is also even they will also break in half, and so on, until the pattern repeats. On examination it appears there are three different cases:

$D = 2^r$; eventually $\alpha = 0$,

$D = 2^r 3$; eventually the solution set is 2^r copies of the SU(2) algebra,

and $D = 2^r K$, where K is odd and $K \neq 3$; eventually we obtain 2^r rings of length K which cycle in just the way described above for D odd. Note that when a ring breaks in two under the commutator mapping the two new rings are not equivalent to the previous one. Basically, this is because no permutation of the γ's can change the length of a ring. Further solutions obtained by considering other subsets of the Clifford algebra have not been investigated fully, but probably have similar properties.

For the case D = 3, we can define the commutator mapping to be

$$\alpha_i' = -i \ \epsilon_{ijk} \alpha_j \alpha_k \tag{2.6}$$

which always produces a new set of three matrices and, in this case, we can also keep track of the signs in the commutators. Clearly, the SU(2) algebra is a fixed point of (2.6). It is also convenient sometimes to write,

$$\alpha_\pm = \alpha_1 \pm \alpha_2, \qquad \alpha_o = \alpha_3, \tag{2.7}$$

in which case the equations (1.1) with $\sigma = 2$, and the mapping (2.6) become

$$\begin{aligned} 4\alpha_o &= [\alpha_+ \ [\alpha_-, \alpha_o]] + [\alpha_-, [\alpha_+, \alpha_o]] \\ 4\alpha_+ &= [\alpha_+, [\alpha_-, \alpha_+]] + 2[\alpha_o, [\alpha_o, \alpha_+]] \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \alpha_o' &= \tfrac{1}{2} [\alpha_+, \alpha_-] \\ \alpha_\pm' &= \pm [\alpha_o, \alpha_\pm], \end{aligned} \tag{2.9}$$

respectively. In the next section we shall consider solutions to (1.1) for D = 3 and N = 2,3.

3 . Special cases when D = 3

N = 2 Here the equation can be solved completely by setting

$$\alpha_\ell = A_{\ell k} \frac{\sigma_k}{2} , \tag{3.1}$$

in which case the cubic equations become,

$$A A^T A = A \, (\mathrm{tr}(A^T A) - 2),$$

with the solutions

$$A \sim \pm 1 \text{ or } \mathrm{diag}\, (\sqrt{2}, \sqrt{2}, 0). \tag{3.2}$$

The first pair of solutions is just ± the SU(2) algebra, the second is inequivalent to these. For (3.2) the invariants (2.2) take the values

$$(q, t) = \left(\frac{3}{2} , \pm \frac{3}{2} \right), \quad (2, 0) \tag{3.3}$$

respectively. Under the commutator mapping the ± SU(2) algebra is a fixed point (or, more precisely, the SU(2) algebra is the only fixed point; the other, - SU(2) algebra is replaced by the SU(2) algebra under the mapping), the other solution is singular - its first image under the mapping is a solution for $\sigma = 0$, the second image is zero.

N = 3 An expansion analogous to eq(3.1) is not very helpful in this case, and indeed we do not yet know if all the solutions are contained in the list outlined below. We can start by pointing out that there are two inequivalent embeddings or representations of the SU(2) algebra in SU(3). One of them, under which the adjoint representation of SU(3) breaks into

$$\underline{8} = \underline{1} + \underline{3} + \underline{2} + \underline{2} \tag{3.4}$$

is merely a repeat of (3.2). The other, under which the adjoint of SU(3) breaks into

$$\underline{8} = \underline{3} + \underline{5} \tag{3.5}$$

is different but nevertheless leads to a triple of solutions like (3.2) for which

$$(q, t) = (6, \pm 6) , \quad (8, 0), \tag{3.6}$$

respectively.

To go further we shall need a representation of the SU(3) generators in which the decomposition (3.5) may be expressed conveniently. Picking a pair of simple roots, normalised so that $\alpha^2 = \beta^2 = 1$, $\alpha \cdot \beta = -\frac{1}{2}$ we may write,

$$\begin{aligned} J_o &= 2(\underline{\alpha} + \underline{\beta}) \cdot \underline{H} \\ J_1 &= \sqrt{2}(E_\alpha + E_\beta) = J_{-1}^\dagger \end{aligned} \tag{3.7}$$

$$\begin{aligned} M_2 &= E_{\alpha+\beta} = (M_{-2})^\dagger \\ M_1 &= -\frac{1}{\sqrt{2}}(E_\alpha - E_\beta) = -(M_{-1})^\dagger \\ M_o &= \frac{2}{\sqrt{6}}(\underline{\alpha} - \underline{\beta}) \cdot \underline{H} \ . \end{aligned} \tag{3.8}$$

Then, the generators J_o, $J_{\pm 1}$ close on the so(3) algebra, whilst the free M's correspond to a quintet with respect to this so(3) subalgebra. Moreover, we also have for the non zero commutators,

$$\begin{aligned} [M_2, M_{-2}] &= J_o, & [M_1, M_{-1}] &= -\tfrac{1}{2} J_o \\ [M_o, M_1] &= -\tfrac{1}{2}\frac{\sqrt{3}}{2} J_1, & [M_2, M_{-1}] &= -\tfrac{1}{2} J_1 \end{aligned} \tag{3.9}$$

and

$$[J_o, M_k] = kM_k, \quad [J_{\pm 1}, M_k] = \sqrt{(2 \mp k)(3 \pm k)} \quad M_{k\pm 1}. \tag{3.10}$$

We give them all in case anyone wants to check. Notice that commutators of M's give J's, which is what we expect - knowing already that SU(3)/SO(3) is a symmetric space. The algebra of the J's and M's is nicely graded, but to obtain a new solution we need to break the grading.

Suppose we set

$$\alpha_\pm = J_{\pm 1}, \qquad \alpha_o = \lambda J_o + \mu(M_2 + M_{-2}), \tag{3.11}$$

then the cubic equations (2.8) are satisfied if $\lambda^2 + \mu^2 = 1$. However, λ and μ are not merely artifacts of the symmetry (2.1) since the invariants come out to be,

$$(q, t) = (6, 6\lambda). \tag{3.12}$$

Hence, as λ varies between ± 1 the solution (3.11) interpolates the maximal embeddings with invariants $(6, \pm 6)$. It is the first example of a continuous family of solutions whose members are not related by the symmetry (2.1). We are confident, on the basis of numerical work, that the one parameter freedom we have found is all there is for this solution. It has precisely twelve degrees of freedom, the parameters of SO(3) × SU(3) together with λ.

If we set

$$\alpha_\pm = \frac{1}{\sqrt{3}} J_{\pm 1} \quad , \quad \alpha_0 = \frac{\sqrt{10}}{3} M_0 \tag{3.13}$$

we obtain another, this time isolated, solution for which the invariants are

$$(q, t) = (\frac{22}{9}, 0).$$

With respect to the commutator mapping the two solutuions (3.12), (3.13) behave in a very different way. Consider (3.12) first. Putting $\lambda = \cos\theta$ and computing the commutators gives

$$\begin{aligned} \alpha_o' = J_o \, , \quad \alpha_\pm' &= \lambda J_{\pm 1} \mp 2\mu M_{\mp 1} \\ &= U(-\theta)\, J_\pm\, U(\theta) \end{aligned} \tag{3.14}$$

where

$$U(\theta) = e^{\theta(M_2 - M_{-2})} \, .$$

Hence, conjugating by $U(\theta)$ we find

$$\alpha_\pm' \sim J_\pm \, , \qquad \alpha_o' \sim \cos 2\theta \, J_o - \mathrm{Sin}\, 2\theta \, (M_2 + M_{-2})$$

which is just the same as (3.11) with θ replaced by -2θ. In other words, the commutator mapping applied to a solution with invariants $(6, 6\cos\theta)$ produces a new one with invariants $(6, 6\cos 2\theta)$. These are only the same provided $\cos\theta = 1$ or $-\frac{1}{2}$, (i.e. $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$), the fixed points of the mapping

$$z \to z' = \frac{1}{z^2} \, . \tag{3.15}$$

The fixed point $\theta = 0$ corresponds of course to the maximal embedding itself. However, the other two fixed points do not correspond to algebras, and are in fact only fixed points up to a conjugation - like the γ-rings of odd length described in section 2.

As for (3.13), the commutator map applied to this one does not produce a new solution. Indeed, repeated application of the map leads eventually to zero, after infinitely many steps, without once visiting another solution on the way. It is rather unique in this respect. After all the examples given so far one might have suspected that the commutator mapping always produced a new solution. However, it is not so, as (3.13) demonstrates.

Unfortunately, we do not yet know if the above solutions are the complete set for N = 3. Apart from the obvious embeddings of these solutions we have found no others for N > 3. However, that is no proof that there are indeed no more. That there are no more might be explained if we could find a definite connection between symmetric spaces and the kinds of solution (3.11), (3.13) displayed above. Apart from some empirical evidence in favour of such a connection for D > 3, there is nothing concrete we can say at the moment.

4. Motivation

In this last section I promised to outline our motivation in considering the cubic algebra. It is connected with the study of Yang-Mills equations in one dimension. More precisely, let T_μ, $\mu = 1,2,3$, or 4, be an SU(N) gauge potential and $T_{\mu\nu}$ its associated field strength. Thus,

$$T_{\mu\nu} = \partial_\mu T_\nu - \partial_\nu T_\mu + [T_\mu, T_\nu] \tag{4.1}$$

$$(T_\mu)^+ = -T_\mu,$$

transforming under gauge transformations as

$$T_\mu \rightarrow T_\mu' = g^{-1} T_\mu g + g^{-1} \partial_\mu g$$

$$T_{\mu\nu} \rightarrow T_{\mu\nu}' = g^{-1} T_{\mu\nu} g,$$

respectively, where g is a position dependent element of SU(N). The field equations satisfied by $T_{\mu\nu}$ are

$$\partial_\mu T_{\mu\nu} + [T_\mu, T_{\mu\nu}] = 0. \tag{4.2}$$

Suppose also, that T_μ, $T_{\mu\nu}$ are functions only of, say, $x^4(\equiv z)$ in the 'gauge' $T_4 = 0$, then the equations (4.2) collapse to the set

$$\frac{d^2T_i}{dz^2} + [T_j, [T_j, T_i]] = 0$$

$$\left[\frac{dT_i}{dz}, T_i\right] = 0. \tag{4.3}$$

These equations have been studied recently by two different sets of people for very different reasons. Firstly, the full equations are interesting as Yang-Mills 'mechanics' - (supposing that z is a time variable, and not Euclidean as written in (4.3)), and display chaotic behaviour (NIKOLAEVSKII and SCHUR, 1982, SAVVIDDY 1983, CHANG 1984). On the other, hand the self-dual equations

$$\frac{dT_i}{dz} = \pm\epsilon_{ijk} T_j T_k \tag{4.4}$$

are interesting because they form the basis of Nahm's construction of the BPS monopoles in three (real) space dimensions (NAHM 1981, 1982, 1983; CORRIGAN and GODDARD 1984). It is crucial, in the latter use, that z be restricted to a finite interval (say, $|z| \leq \frac{1}{2}$, if the BPS monopoles are those corresponding to a broken SU(2) gauge theory in three dimensions), that $\underline{T}$ has the form

$$\underline{T} \sim \frac{i\underline{\alpha}}{\frac{1}{2} \pm z} , \tag{4.5}$$

at the ends, and is otherwise non singular throughout the interval. We wondered to what extent it would be possible to obtain solutions to the second order equations (4.3), also on a finite interval, maintaining the simple pole behaviour at the ends. On that basis, the pole residue $\underline{\alpha}$ must satisfy the cubic algebra (1.1), as is easily seen substituting eq(4.5) into (4.3), and $\underline{T}(z)$ must interpolate between two such poles. In the preceding sections we have made some remarks about the possible pole residues but we still have no clear idea in what combinations they may be assembled to yield a full solution - that seems to be a difficult question. We further wondered if these solutions might play a role in discovering the elusive non dual monopoles whose existence has been proved by Taubes (TAUBES 1982).

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