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THE INDEX OF TRANSVERSALLY ELLIPTIC COMPLEXES

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Let M be a closed smooth manifold and $[T^*M, \pi, M]$ be the cotangent bundle of M with projection π: $T^*M \longrightarrow M$. By $T'M$ we will denote the set of non-trivial covectors on M.

<u>Definition 1:</u> A (finite) set $\underline{E}$ of complex vector bundles $[E_j, p_j, M]$, $j=0,\dots,N$, of finite rank on M and differential operators D_j: $\Gamma(E_j) \longrightarrow \Gamma(E_{j+1})$, $j=0,\dots,N-1$, is called a <u>complex</u> iff

$$D_{j+1} \circ D_j = 0 \qquad j=0,\dots,N-2. \tag{1}$$

Here $\Gamma(.)$ denotes the space of smooth sections, $\Gamma_0(.)$ those with compact support.

In the sequel we will make the basic assumption that all the operators D_j are of order 1. This leads to technical simplifications and is in accordance with most of the examples. Moreover, all vector bundles will be complex and of finite rank.

Studying a complex it is natural to consider its cohomology:

<u>Definition 2:</u> If $\underline{E}$ is a complex on M, then the <u>spaces of cocycles</u> $Z^j(\underline{E})$, <u>coboundaries</u> $B^j(\underline{E})$ and the <u>cohomology spaces</u> $H^j(\underline{E})$ are defined as

$$Z^j(\underline{E}) =_{Df} \ker D_j, \; j=0,\dots,N-1, \qquad Z^N(\underline{E}) =_{Df} \Gamma(E_N),$$
$$B^j(\underline{E}) =_{Df} \operatorname{im} D_{j-1}, \; j=1,\dots,N, \qquad B^0(\underline{E}) =_{Df} \{0\} \text{ and}$$
$$H^j(\underline{E}) =_{Df} Z^j(\underline{E})/B^j(\underline{E}), \; j=0,\dots,N, \text{ respectively.}$$

For $N=1$ we thus have $H^0 = \ker D$ and $H^1 = \operatorname{coker} D$. The fundamental observation of index theory now is that for an elliptic operator D: $\Gamma(E_0) \longrightarrow \Gamma(E_1)$ these cohomology spaces are finite-dimensional, i.e. the index $\operatorname{ind}(D) =_{Df} \dim(\ker D) - \dim(\operatorname{coker} D)$ is a well-defined integer.

For a single differential operator D: $\Gamma(E_0) \longrightarrow \Gamma(E_1)$ ellipticity means that the (principal) symbol morphism $\sigma(D)$: $\pi^*(E_0) \longrightarrow \pi^*(E_1)$ between the bundles which are lifted via π to $T'M$ is an isomorphism.

For a complex this is generalized to the following

Definition 3: The complex $\underline{E}$ on M is called elliptic iff its symbol sequence

$$0 \to \pi^* E_0 \xrightarrow{\sigma(D_0)} \pi^* E_1 \xrightarrow{\sigma(D_1)} \dots \xrightarrow{\sigma(D_{N-1})} \pi^* E_N \to 0$$

is an exact sequence of vector bundle morphisms over T'M, i.e. for each $\xi_x \in T'M$

$$\ker \sigma(D_j)(\xi_x) = \operatorname{im} \sigma(D_{j-1})(\xi_x)\ ,\ j=1,\dots,N-1,$$
$$\ker \sigma(D_0)(\xi_x) = \{0\}\ ,\ \operatorname{im} \sigma(D_{N-1})(\xi_x) = (\pi^* E_N)_{\xi_x}\ . \tag{2}$$

By reduction to the case of single operators one can conclude that ellipticity of a complex implies finite-dimensionality of the cohomology spaces. Hence, for an elliptic complex $\underline{E}$ the index

$$\operatorname{ind}(\underline{E}) =_{Df} \sum_{j=0}^{N} (-1)^j \dim H^j(\underline{E})$$

is a well-defined integer again.
In order to study non-elliptic complexes we introduce a family of sets comprising those covectors for which (2) is violated.
Definition 4: If $\underline{E}$ is a complex on M, then the j-th characteristic set of $\underline{E}$ is defined as the set

$$C_j(\underline{E}) =_{Df} \{\xi_x \in T'M\colon \ker \sigma(D_j)(\xi_x) \supsetneq \operatorname{im} \sigma(D_{j-1})(\xi_x)\}\ ,$$

for $j=1,\dots,N-1$, $C_0(\underline{E}) =_{Df} \{\xi_x \in T'M\colon\ \sigma(D_0)(\xi_x)$ is not injective$\}$,

$$C_N(\underline{E}) =_{Df} \{\xi_x \in T'M\colon\ \sigma(D_{N-1})(\xi_x) \text{ is not surjective}\}.$$

$C(\underline{E}) =_{Df} \bigcup_{j=0}^{N} C_j(\underline{E})$ is called the characteristic set of $\underline{E}$.

Thus, $C(\underline{E})$ contains those covectors, at which the symbol sequence fails to be exact and, hence, $\underline{E}$ is elliptic if and only if $C(\underline{E})=\emptyset$.
For various geometric reasons it is natural to consider a Lie group G acting linearly (from the left) on the bundles E_j such that the induced actions on M coincide.
Definition 5: The complex $\underline{E}$ is called a G-complex iff all the operators D_j, $j=0,\dots,N-1$, are G-equivariant, i.e. iff they commute with the induced actions on the corresponding $\Gamma(E_j)$.

The Lie group G will be assumed to be countable at infinity.
For a G-complex the cohomology spaces are representation spaces for G. So, for an elliptic G-complex $\underline{E}$ the G-index is defined as the alternating sum of these finite-dimensional representations

$$\operatorname{ind}_G(\underline{E}) =_{Df} \sum_{j=0}^{N} (-1)^j\ r_{H^j(\underline{E})}, \tag{3}$$

where the r.h.s. can be understood either as an element of the representation ring R(G) or as the corresponding smooth function $\sum_{j=0}^{N}(-1)^j\chi_j$, χ_j being the character of $r_{H^j(\underline{E})}$, $\chi_j(g)=\text{trace } r_{H^j}(g)$ for $g\in G$.

In a series of lectures ([1]) M.F.Atiyah and I.M.Singer initiated the study of a class of complexes called "transversally elliptic" which contains the elliptic ones. For these complexes the first interpretation of (3) makes sense even if the cohomology is not finite-dimensional. The existence proof in [1] and the following (mainly K-theoretic) investigations rely heavily on the compactness of the group G. However, in a survey article ([8]) I.M.Singer described an existence proof for the index of a transversally elliptic complex for a not necessarily compact group suggested by L.Hörmander.

It is our aim here to prove this result following the suggestion of Hörmander, i.e. applying elementary notions and facts from microlocal analysis. As basic references for these the reader may consult [3], [4] and, in particular, [6] and [7].

Definition 6: Let the Lie group G act on the manifold M (from the left). The covector $\xi_x\in T'M$ is called transversal w.r.to G iff ξ_x annihilates all vectors tangent to the orbit through x.

In other words $\langle\xi_x,\tilde{Y}(x)\rangle = 0$ for all $Y\in\mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G, $\tilde{Y}$ the fundamental vector field defined by Y and the action and $\langle.,.\rangle$ the natural pairing of vectors and covectors.

Let $T_G'M$ be the set of non-trivial covectors on M which are transversal w.r.to G, $T_G^*M =_{Df} T_G'M \cup \{0_x\in T^*M:\ x\in\pi(T_G'M)\}$.

Remark: If we consider T^*M with its natural symplectic form as a symplectic manifold the induced action of G is Hamiltonian. The corresponding moment map $J:T^*M\longrightarrow\mathfrak{g}^*$ is given by

$$\langle J(\xi_x),Y\rangle_{\mathfrak{g}} =_{Df} \langle\xi_x,\tilde{Y}(x)\rangle \quad \text{for } \xi_x\in T^*M,\ Y\in\mathfrak{g}.$$

Thus, $T_G'M = T'M\cap J^{-1}(0)$. As the moment map is equivariant w.r.to the coadjoint action on $\mathfrak{g}^*$, this implies the G-invariance of $T_G'M$.

Definition 7: The G-complex $\underline{E}$ on M is transversally elliptic iff

$$T_G'M\cap C(\underline{E}) = \emptyset.$$

So, at a covector $\xi_x\in T'M$ which is transversal w.r.to G the symbol sequence of $\underline{E}$ is exact, but in orbit direction there are no conditions except the general G-equivariance of $\underline{E}$.

There are the following extreme cases:

(i) If G is finite or if the action is trivial then $T_G'M = T'M$. In this case transversal ellipticity of a G-complex coincides with its ellipticity.

(ii) If G acts transitively on M then $T_G'M = \emptyset$. Hence, every complex which is equivariant w.r.to a group acting transitively on the base is transversally elliptic.

In the definition of the index for a transversally elliptic complex Atiyah and Singer were guided by the construction of the Harish-Chandra character for representations of semi-simple Lie groups:

Under rather weak assumptions concerning its continuity a representation of a Lie group G lifts to a representation of the convolution algebra $L^1(G)$. In particular the test functions on G act as bounded linear operators in the representation space. Once one can show that these operators are of trace-class, their trace defines a functional which is continuos for quite general reasons. This construction associates with a certain class of infinite-dimensional representations a distribution on the group which generalizes the character of a finite-dimensional representation.

To apply the relevant Hilbert space techniques we need some auxiliary structure:

Let $\underline{E}$ be a G-complex on the closed manifold M. Choose arbitrary Hermitian scalar products on the E_j, $j=0,\dots,N$, and a positive smooth density on M. Then the spaces $L^2(E_j)$ of square-integrable sections are defined and the action of G on $\Gamma(E_j)$ extends to a continuos representation r_j of the Lie group G in the Hilbert space $L^2(E_j)$, $j=0,\dots,N$.

Denoting the distributional or generalized sections of E_j (in the sense of L.Schwartz; cf.[2]) by $D'(E_j)$ we define unbounded operators $\overline{D}_j : L^2(E_j) \longrightarrow L^2(E_{j+1})$, $j=0,\dots,N-1$, with dense domain by their graphs $\{(u,v) \in L^2(E_j) \times L^2(E_{j+1}) : D_j u = v \text{ in } D'(E_{j+1})\}$.

Here we used the natural extension of a differential operator to distributional sections. As can be seen by looking at the graphs the $\overline{D}_j$ are closed. As their adjoints are closed, too, the spaces

$$\underline{H}^j(\underline{E}) = \ker \overline{D}_j \cap \ker \overline{D}_{j-1}^*, \quad j=1,\dots,N-1,$$
$$\underline{H}^0(\underline{E}) = \ker \overline{D}_0, \quad \underline{H}^N(\underline{E}) = \ker \overline{D}_{N-1}^*$$

are closed subspaces of $L^2(E_j)$, $j=0,\dots,N$, which will be considered as a substitute for the cohomology spaces $H^j(\underline{E})$. By P_j we will denote the orthogonal projection onto $\underline{H}^j(\underline{E})$ in $L^2(E_j)$.

Let R_j be the extension of the representation r_j to $L^1(G)$ defined

by $R_j(f) =_{Df} \int_G f(g)\, r_j(g)dg$, where dg is a left-invariant Haar measure on G which will be fixed from now on.
Now we can state the main result.

Theorem: (Atiyah, Singer; Hörmander)
Let $\underline{E}$ be a transversally elliptic G-complex on the closed manifold M, G a Lie group.
Then for each test function $f \in C_0^\infty(G)$ on G the operators

$$P_j \circ R_j(f) \circ P_j : L^2(E_j) \longrightarrow L^2(E_j),\ j=0,\dots,N,$$

are of trace-class and the assignment

$$f \longmapsto \mathrm{trace}(P_j \circ R_j(f) \circ P_j)$$

is a distribution on G.

Once the first statement in the theorem is proved the second follows by quite general arguments involving traces (cf. [5]).
Assuming for a moment the validity of the theorem we can define the index.

<u>Definition 8:</u> Let $\underline{E}$ be a transversally elliptic G-complex on M. The index of $\underline{E}$ is defined to be the distribution $\mathrm{ind}_G(\underline{E})$ on G given by

$$\langle \mathrm{ind}_G(\underline{E}), f \rangle =_{Df} \sum_{j=0}^{N} (-1)^j\ \mathrm{trace}(P_j \circ R_j(f) \circ P_j)$$

for $f \in C_0^\infty(G)$.

<u>Remark:</u> In the case of an elliptic complex Hodge theory identifies $\underline{H}^j(\underline{E})$ and $H^j(\underline{E})$, the distribution $\mathrm{ind}_G(\underline{E})$ is smooth and coincides with the G-index defined by (3).

The proof of the theorem makes use of three basic facts:

(i) The inclusions of the smooth sections into the distributional sections given by the Hermitian metrics and the measure on the base manifold M $\Gamma(E_j) \longrightarrow D'(E_j)$, $j=0,\dots,N$, are of trace-class, because M is closed.

(ii) A microlocal regularity result for differential operators.

(iii) A sufficient condition for the existence of the composition of operators with distribution kernel together with a formula for the behaviour of the wave front relation.

In the following we will drop the index j as the argument is the same for each value of j.

The application of (i) is immediate: It is enough to show that the operator $R(f) \circ P$ maps $L^2(E)$ into $\Gamma(E)$ so that it factors through the inclusion. The necessary continuity property follows from the closed graph theorem, and by the commutativity of the trace we have trace($P\circ R(f)\circ P$) = trace($R(f)\circ P^2$) = trace($R(f)\circ P$).

In order to show that $R(f)\circ P(u)\in \Gamma(E)$ for every $u\in L^2(E)$ Hörmander suggested to use the wave front set of a distribution. We recall the relevant definitions.

<u>Definition 9:</u> Consider a vector bundle F on the manifold Y. Assume that a positive smooth density is chosen on Y and that F is equipped with a Hermitian metric.
Let $u\in D'(F)$ be a distributional section of F.
The point $x_0\in Y$ does not belong to the <u>singular support</u> of u, $x_0\notin$ sing supp(u), iff there is a function $f\in C^\infty(Y)$, $f(x_0)\neq 0$, for which fu is smooth, i.e. $fu\in\Gamma(F)\subset D'(F)$.
The covector $\xi_x\in T'Y$ does not belong to the <u>wave front set</u> of u, $\xi_x\notin WF(u)$, iff for each function $\Psi\in C_0^\infty(Y\times\mathbb{R}^p;\mathbb{R})$, $(d\Psi(.,a))_x=\xi_x$, there is an $s\in\Gamma(F)$, $s(x)\neq 0$, and a neighbourhood A of a in $\mathbb{R}^p$, such that for all $n\in\mathbb{N}$

$$\left|\langle u,\exp(-it\Psi(.,a'))\, s\rangle\right| = O(t^{-n})$$

for $a'\in A$ and $t\to\infty$.

While the singular support of a distributional section is essentially clear by itself, the wave front set may need some explanation. For a motivation by means of the Paley-Wiener theorem we refer to [3]. Instead of this we list a few properties which are immediate consequences of the definition.

- The wave front set of a distributional section $u\in D'(F)$ is a closed subset of T'Y which is invariant under multiplication by positive real numbers.
- The projection of the wave front set on the base is equal to the singular support, $\pi(WF(u))$ = sing supp(u).

Thus, in order to prove that a distributional section $u\in D'(F)$ is smooth it suffices to show that $WF(u)=\emptyset$.
The following rather simple kind of distributional sections will turn out to be useful later.

<u>Definition 10:</u> Let F be a vector bundle on the manifold Y without boundary. A submanifold X of Y without boundary together with a fixed positive smooth density $d\nu$ and a smooth section s of the dual bundle $F^*|_X=(F|_X)^*$ define a distributional section u(X,s) of F by

$$\langle u(X,s),t\rangle =_{Df} \int_X \langle s(x),t(x)\rangle_x\, d\nu \quad \text{for } t\in\Gamma(F) \qquad (4)$$

if X is compact or s has compact support.

We will call u(X,s) the <u>geometric current</u> on F defined by X and s.

Locally a geometric current is just the tensor product of a delta-distribution and a smooth function. This implies the following:
- If u(X,s) is the geometric current defined by X and s, then
  $WF(u(X,s)) = N_Y^{\cdot}(X)|_{supp(s)} = \{\xi_x \in T'Y:\ x \in supp(s),\ \xi_x|_{T_xX}=0\}$,
  i.e. the wave front set of a geometric current is contained in the conormal bundle of its underlying manifold minus the zero section.

Now we can formulate the facts (ii) and (iii).

(ii) Microlocal regularity of differential operators
Let E and F be vector bundles on the closed manifold Y and $D: \Gamma(E) \longrightarrow \Gamma(F)$ be a differential operator (which will not be distinguished from its extension to distributional sections, $D: D'(E) \longrightarrow D'(F)$).
If $C_o(D)$ denotes as before the set of non-trivial covectors $\xi_x \in T'M$ such that $\sigma(D)(\xi_x)$ is not injective (cf. Definition 4), then

$$WF(Du) \subset WF(u) \subset WF(Du) \cup C_o(D), \quad u \in D'(E).$$

For a proof we refer to [3], [6], [7] or [9].
If, in particular, Du=0 for $u \in D'(E)$ this yields $WF(u) \subset C_o(D)$.

For the formulation of fact (iii) we still need some preparation. Let X, Y be smooth manifolds without boundary with fixed positive smooth densities and E, F be vector bundles on X and Y, respectively, which are equipped with Hermitian metrics.
Consider a continuos linear operator $A: \Gamma_o(E) \longrightarrow D'(F)$, then, by the Schwartz' Kernel Theorem, there is a uniquely determined kernel $K_A \in D'(F \boxtimes E)$, which is a distributional section of the exterior tensor product of F and E on Y x X, such that for all $s \in \Gamma_o(E)$, $t \in \Gamma_o(F)$ $\langle As,t\rangle = \langle K_A, t \boxtimes s\rangle$.

<u>Definition 11:</u> Let $A: \Gamma_o(E) \longrightarrow D'(F)$ be linear and continuos and $K_A \in D'(F \boxtimes E)$ be its distributional kernel. The <u>wave front relation</u> of A and its projections are defined as the sets

$WF'(A) =_{Df} \{(\eta_y, \xi_x) \in T'(Y \times X):\ (\eta_y, -\xi_x) \in WF(K_A)\}$,

$WF_X'(A) =_{Df} \{\xi_x \in T'X:$ there is a point $y \in Y$ such that $(0_y, \xi_x) \in WF'(A)\}$,

$WF'_Y(A) =_{Df} \{\eta_y \in T'Y$: there is a point $x \in X$ such that $(\eta_y, 0_x) \in WF'(A)\}$.

(iii) Composition formula for operators with distributional kernel
Let X, Y, Z be smooth manifolds without boundary with fixed smooth positive densities and E, F, H be vector bundles on X, Y, Z, respectively, equipped with Hermitian metrics.
Consider continuos linear operators $A: \Gamma_0(F) \longrightarrow D'(E)$ and $B: \Gamma_0(H) \longrightarrow D'(F)$.
If $WF'_Y(A) \cap WF'_Y(B) = \emptyset$ and the projection from the set $\{(x,y,z) \in X \times Y \times Z: (x,y) \in \mathrm{supp}(K_A), (y,z) \in \mathrm{supp}(K_B)\}$ to $X \times Z$ mapping (x,y,z) to (x,z) is proper, then there is a well-defined composition $A \circ B: \Gamma_0(H) \longrightarrow D'(E)$ which is linear and continuos.
Moreover,

$$\mathrm{supp}(K_{A\circ B}) \subset \mathrm{supp}(K_A) \circ \mathrm{supp}(K_B)$$

and

$$WF'(A\circ B) \subset WF'(A) \circ WF'(B) \cup [WF'_X(A) \times 0_Z] \cup [0_X \times WF'_Z(B)]. \quad (5)$$

In either formula the composition on the r.h.s. is the usual composition of relations and 0_X, 0_Z denote the zero section in T^*X and T^*Z, respectively. For a proof see [3], [4] or [6].

<u>Corollary:</u> Let $A: \Gamma_0(F) \longrightarrow D'(E)$ be as above. Then A can be continuosly extended to the set of those $u \in D'(F)$ with compact support, for which $WF(u) \cap WF'_X(A) = \emptyset$.
Moreover, under this condition,

$$WF(Au) \subset WF'(A) \circ WF(u) \cup WF'_Y(A).$$

Now we return to the situation we are interested in:
$\underline{E}$: $0 \longrightarrow \Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{N-1}} \Gamma(E_N) \longrightarrow 0$ is a transversally elliptic G-complex on the closed manifold M.
$R_j(f) \circ P_j$ is the composition of the orthogonal projection

$$P_j: L^2(E_j) \longrightarrow \underline{H}^j(\underline{E})$$

and the operator $R_j(f) = \int_G f(g)\, r_j(g)\, dg,\ f \in C_0^\infty(G)$.

<u>Lemma 1:</u> If $u \in \underline{H}^j(\underline{E})$, then $WF(u) \subset C_j(\underline{E})$.
Proof: Fix $j \in \{1,\ldots,N-1\}$. Then $u \in \underline{H}^j(\underline{E})$ implies that $\overline{D}_j u = 0 = \overline{D}^*_{j-1} u$. Then we know by (ii) that $WF(u) \subset C_0(\overline{D}_j) \cap C_0(\overline{D}^*_{j-1})$, but the latter is equal to $C_j(\underline{E})$ by definition. For j=0 or N this is immediately seen.

Next we observe that $R_j(f)$ is the composition of two operators which have a rather simple shape. As before we omit the index j.
$R(f) = B \circ A$, where $A: \Gamma(M;E) \longrightarrow \Gamma_0(G \times M; pr_2^* E)$,

$$(As)(g,x) =_{Df} f(g)\,(r(g)s)(x)$$

and

$$B: \Gamma_0(G \times M; pr_2^* E) \longrightarrow \Gamma(M;E),$$
$$(Bt)(x) =_{Df} \int_G t(g,x)\,dg.$$

Now we are going to describe the distributional kernel of A and B in order to compute their wave front relation.

<u>Lemma 3:</u> The distributional kernel of A is the geometric current defined by the submanifold

$$Y = \{(g,x,g^{-1}x):\ g \in G,\ x \in M\}$$

of G x M x M and the section a of the bundle obtained by lifting $E^* \boxtimes E^*$ to G x M x M and restricting to Y,
$\langle a(g,x,g^{-1}x), e \boxtimes e' \rangle = (e, \overline{ge'})_E$ for $e \in E_x$, $e' \in E_{g^{-1}x}$.
The density on Y is obtained from the one on G x M via the diffeomorphism $(g,x) \in G \times M \longmapsto (g,x,g^{-1}x) \in Y$.

This is an immediate consequence of Definition 10.

Lemma 3 implies that $\mathrm{supp}(K_A) \subset \{(g,x,g^{-1}x) \in Y:\ g \in \mathrm{supp}(f)\}$ and, hence, is compact. Moreover, the wave front relation of A is contained in the conormal bundle of Y in G x M x M.
Using the moment map the latter has a very simple description:
$WF'(A) \subset \{(\gamma_g, \xi_x, g^{-1}\xi_x) \in T^*G \times T'M \times T'M:\ J(g^{-1}\xi_x) = g\,\gamma_g,\ g \in \mathrm{supp}(f)\}$.
On the r.h.s. of the equality the G-action is the canonical one on T^*G.

<u>Corollary:</u> The projections of the wave front relation of A on both components are empty, $WF'_M(A) = WF'_{G \times M}(A) = \emptyset$.

<u>Lemma 4:</u> The distributional kernel K_B of B is the geometric current on M x G x M defined by the submanifold

$$Z = \{(x,g,x):\ x \in M,\ g \in G\}$$

and the section b of the bundle $pr_1 E^* \boxtimes pr_3 E^*$ restricted to Z, $\langle b(x,g,x), e \boxtimes e' \rangle = (e, \bar{e}')_E$ for $e \in E_x$, $e' \in E_x$.
The density on Z is obtained from the one on G x M via the diffeomorphism $(g,x) \in G \times M \longrightarrow (x,g,x) \in Z$.

Again this can be seen by checking the definition of the distributional kernel.

<u>Corollary:</u> $WF'(B) \subset \{(\xi_x, 0_g, \xi_x) \in T'M \times T^*G \times T'M\}$ and
$WF'_M(B) = WF'_{G\times M}(B) = \emptyset$.

Applying (iii) to A and B we obtain from Lemma 3 and 4 together with their corollaries

$$WF'(R(f)) \subset WF'(B) \circ WF'(A) \subset \{(\xi_x, g^{-1}\xi_x):\ J(g^{-1}\xi_x) = 0,\ g \in \mathrm{supp}(f)\}$$
$$= \{(\xi_x, g^{-1}\xi_x):\ \xi_x \in T'_GM,\ g \in \mathrm{supp}(f)\}.$$

In particular this implies that both projections of the wave front relation of R(f) are empty.

Consider now a square-integrable section u of E_j, $u \in L^2(E_j)$. From Lemma 1 we conclude that $WF(P_j(u)) \subset C_j(\underline{E})$. As we know that both projections of the wave front relation of $R_j(f)$ are empty and M is compact the corollary of (iii) implies that $R_j(f) \circ P_j(u)$ is a well-defined distributional section of E_j. Moreover, we obtain

$$WF(R_j \circ P_j(u)) \subset WF'(R_j(f)) \circ WF(P_j(u)) \subset$$
$$\subset \{(\xi_x, g^{-1}\xi_x):\ \xi_x \in T'_GM,\ g \in \mathrm{supp}(f)\} \circ C_j(\underline{E}) = \emptyset$$

as the complex $\underline{E}$ was assumed to be transversally elliptic, i.e. $T'_GM \cap C_j(\underline{E}) = \emptyset$ for all $j=0,\dots,N$.
Thus we proved $WF(R_j \circ P_j(u)) = \emptyset$, i.e. $R_j \circ P_j(u) \in \Gamma(E_j)$.

This is what we wanted to show.

REFERENCES

[1] ATIYAH M.F. "Elliptic operators and compact groups", Lect. Notes Math. 401/Springer-Verlag/Berlin/1974.

[2] ATIYAH M.F., BOTT R. "A Lefschetz fixed point formula for elliptic complexes.I", Ann. of Math./ <u>86</u> /1967/, 374-407.

[3] DUISTERMAAT J.J. "Fourier Integral Operators", Courant Inst./ New York/1973.

[4] GABOR A. "Remarks on the wave front set of a distribution", Trans. Amer. Math. Soc./ <u>170</u> /1972/, 239-244.

[5] GEL'FAND I.M., VILENKIN N.YA. "Generalized Functions", Vol. 4/ Moscow/1961.

[6] HÖRMANDER L. "Fourier integral operators", Acta Math./ <u>127</u> /

1971/, 79-183.

[7] HÖRMANDER L. "The Analysis of Linear Partial Differential Operators.I", Springer-Verlag/Berlin/1983.

[8] SINGER I.M. "Recent applications of index theory for elliptic operators", Proc. Symp. Pure Math./ 23 /1973/, 11-31.

[9] ŠUBIN M.A. "Pseudodifferential operators and spectral theory", Moscow/1978.

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