Robert A. Wolak

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ON TRANSVERSE STRUCTURES OF FOLIATIONS

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Many authors have considered geometrical structures on the normal bundle of a foliation. It is natural to consider only these structures which are parallel along the leaves of the foliation or as some authors say projectible. As examples we can mention bundle-like metrics, transversely projectible or basic connections, transverse symplectic structures, and in general transverse G-structures. Various properties of these structures have been shown, very often similar to those well known for corresponding structures on manifolds. In this paper we shall endeavour to show how to obtain such results in a most general way.

Let M be a smooth manifold of dimension n, and F a codimension q foliation on M defined by a cocycle $\{U_i, f_i, g_{ij}\}$ where $\{U_i\}$ form an open covering of M, $f_i : U_i \longrightarrow R^q$ is a submersion, and $g_{ij} : f_j(U_i \cap U_j) \longrightarrow f_i(U_i \cap U_j)$ is a diffeomorphism such that $f_j \mid U_i \cap U_j = g_{ji} f_i \mid U_i \cap U_j$. Let M_F be a smooth q dimensional manifold equal to $\bigsqcup_i f_i(U_i)$. Then the mappings g_{ij} can be considered as local diffeomorphisms of the manifold M_F and the foliation F as modelled on M_F. If the manifold M is compact we can take a finite set of indices.

Analogously as normal bundles of order r (cf. [7]), we can define transverse (p,r)-velocities and transverse A-bundles.

Example 1. Transverse (p,r)-velocities (p^r-jets).

Let m be a point of the manifold M. Let $f : (R^p, 0) \longrightarrow (M, m)$

be any local smooth mapping of R^p mapping 0 into m . Let f,g be two such mappings and let (U,φ) be an adapted chart such that $\varphi : U \longrightarrow R^{n-q}xR^q$, $\varphi(x) = (\varphi_1(x), \varphi_2(x))$, thus φ_2 is constant along the leaves. We shall also use the notation $\varphi_1 = (y_1,\dots,y_{n-q})$, $\varphi_2 = (x_1,\dots,x_q)$. We say that the mappings f,g are equivalent if $j^p_0 \varphi_2 g = j^p_0 \varphi_2 f$. This is equivalent to

$$\partial^{|v|}/\partial x^v(x_i f) = \partial^{|v|}/\partial x^v(x_i g) \quad \text{for any multiindex } v \in N^p, |v| \leqslant r,$$

$i = 1,\dots,q$.

We shall denote the number of such indices by $p(r)$ and the set of such indices by $N(p,r)$. This equivalence relation does not depend on the choice of an adapted chart at the point m . The equivalence class of a mapping f we denote by $[f]^r_p$. The set of all equivalence classes at a point m we denote by $N^{p,r}_m(M,F)$, and the space $\bigcup_{m\in M} N^{p,r}_m(M,F)$ by $N^{p,r}(M,F)$. By π^r_p let us denote the natural projection of $N^{p,r}(M,F)$ into M , i.e. $\pi^r_p([f]^r_p) = f(0)$. One can easily check that for any adapted chart (U,φ) the set $\bigcup_{m\in U} N^{p,r}_m(M,F)$ is isomorphic to $UxR^{q\cdot p(r)}$ and that the isomorphism is given by the mapping $[f]^r_p \longmapsto (\partial^{|v|}/\partial x^v (x_i f))^v_{i=1,\dots,q}$

Thus, if we denote the mapping defined above by φ^r_p , $\varphi^r_p : (\pi^r_p)^{-1}(U) \longrightarrow R^{n-q}xR^q xR^{q\cdot p(r)}$, the collection of all such φ^r_p defined by an adapted atlas on M , defines an atlas on the space $N^{p,r}(M,F)$. To see that, one has only to notice that if φ_i , φ_j are two adapted charts for a foliated manifold (M,F) , the composition $\varphi_i \varphi_j^{-1} : R^{n-q}xR^q \longrightarrow R^{n-q}xR^q xR^q$ is of the form $(f_1(y,x) , f_2(x))$, where y denotes the first n-q coordinates, x the last q , $f_1 : R^{n-q}xR^q \longrightarrow R^{n-q}$, and $f_2 : R^{n-q}xR^q \longrightarrow R^q$, then

$\varphi^{r}_{ip}(\varphi^{r}_{jp})^{-1} : R^{n-q}xR^{q}xR^{q\cdot p(r)} \longrightarrow R^{n-q}xR^{q}xR^{q\cdot p(r)}$ is equal to

$(f_1, T^{r}_{p}(f_2))$, where $T^{r}_{p}(f_2)$ is the mapping of $T^{r}_{p}(R^{q}) = R^{q}xR^{q\cdot p(r)}$ induced by f_2.

Summing up, we have proved that $N^{p,r}(M,F)$ is locally trivial fibre bundle, whose total space admits a codimension $q.p(r)+q$ foliation F^{r}_{p} projecting by π^{r}_{p} onto the initial foliation F.

If $p=n$ and we take only local diffeomorphisms of R^{n} into M, the above construction gives a bundle called the transverse frame bundle of the foliated manifold (M,F) and denoted by $L^{r}(M,F)$, which is a principal fibre bundle with the fibre L^{r}_{q}.

Example 2. The bundle of transverse A-points of (M,F).
Let A be an associative algebra over the field R with the unit 1. The algebra A is called local if it is commutative, of finite dimension over R, and if it admits the unique maximal ideal $\mathfrak{m}$ of codimension 1 such that $\mathfrak{m}^{h+1} = 0$ for some non-negative integer h. The smallest such an h is called the height of A. Let $R[p] = R[x_1,\dots,x_p]$ be the algebra of all formal power series in $x_1,\dots,x_p$, and let $\mathfrak{m}_p$ be the maximal ideal of $R[p]$ of all formal power series without constant terms. Let $\mathfrak{a}$ be a non-trivial ideal of $R[p]$ such that $R[p]/\mathfrak{a}$ is of finite dimension. Then $A = R[p]/\mathfrak{a}$ is a local algebra with the maximal ideal $\mathfrak{m} = \mathfrak{m}_p/\mathfrak{a}$. Any local algebra is isomorphic to such a local algebra (cf.[3]).
Let $C^{\infty}_{m}(M,F)$ be the algebra of the germs of smooth functions constant of the leaves of the foliations F. An algebra homomorphism $\mathcal{L} : C_{m}(M,F) \longrightarrow A$ will be called an A-point of (M,F) near to m / or infinitely near transverse point to m of kind A / if $\mathcal{L}(f) \equiv f(m) \bmod \mathfrak{m}$ for every $f \in C^{\infty}_{m}(M,F)$. We denote by $A_{m}[M,F]$ the set of all A-points of (M,F) near to m and by $A[M,F] = \bigcup_{m\in M} A_{m}[M,F]$. The mapping $A_{m}[M,F] \ni \mathcal{L} \longmapsto m \in M$ is denoted by π_{A}.
Let $A = R[p]/\mathfrak{a}$, $\mathfrak{m} = \mathfrak{m}_p/\mathfrak{a}$ and $p_A : R[p] \longrightarrow A$ be the natural projection. Let us denote by N the dimension of $\mathfrak{m}$. Let $\tau : C^{\infty}_{0}(R^{p}) \longrightarrow R[p]$ be the natural mapping, where $C^{\infty}_{0}(R^{p})$ denotes

the set of germs of smooth functions on R^p at 0.

Definition. Let f,g be two smooth mappings of R^p into M such that $f(o) = g(o) = m$. We shall say that f is A-equivalent to g at m if $\tau(hf) = \tau(hg) \bmod \mathfrak{m}$ for any $h \in C^\infty_m(M,F)$. By $[f]_A$ we denote the equivalence class of f, by $A_m(M,F)$ all equivalence classes at m, and $A(M,F) = \bigcup_{m\in M} A_m(M,F)$.

Let $[f]_A \in A_m(M,F)$. then $\alpha_f(h) = p_A(hf)$ is an A-point of (M,F) near m, where $h \in C^\infty_m(M,F)$. This correspondence $A(M,F) \ni [f]_A \longmapsto \alpha_f \in A[M,F]$ is a bijection. The proof of this is the same as of Lemma 1.8 of [3].

Let $(U,\varphi) = (U,(y_1,\dots,y_{n-q},x_1,\dots,x_q))$ be an adapted chart, let $b_1,\dots,b_N$ be a basis of $\mathfrak{m}$. On the set $\pi_A^{-1}(U)$ we can define the following chart:

Let α be an A-point of (M,F) near to (y,x). Then $\alpha(x_i) = \sum_{k=1}^{N} a_i^k b_k + x_i$. Put

$$\varphi_A(\alpha) = (y,x,(a_i^k)^{k=1,\dots,N}_{i=1,\dots,q}) .$$

Using the same methods as in Lemma 1.9 and 1.10 of [3] we can check that φ_A is bijective.

Let (U,φ) and (V,ψ) be two adapted charts and let

$\varphi\circ\psi^{-1} : \psi(U\cap V) \longrightarrow \varphi(U\cap V)$ be of the form (f_1,f_2)

where $f_1: R^n \longrightarrow R^{n-q}$, $f_2 : R^q \longrightarrow R^q$. Then $\varphi_A\circ\psi_A^{-1} = (f_1,A(f_2))$/ for the definition of $A(f)$ see [3]/. Let $\mathcal{U} = (U_i,\varphi_i)$ be an adapted atlas of the manifold (M,F). With the differentiable structure defined by $(\pi_A^{-1}(U_i),\varphi_{iA})$, the set $A[M,F]$ is a smooth manifold, $\pi_A: A[M,F] \longrightarrow M$ a fibre bundle over M with the fibre A. On the manifold $A[M,F]$ there is a canonically defined foliation F_A of the same dimensional as the foliation F. The projection π_A maps leaves of the foliation F_A onto leaves of the foliation F . The bundle admits a global section, the zero section s_o. For any transverse mapping $f : M_1 \longrightarrow M_2$ to the foliation F of the manifold M_2 , f defines a smooth mapping $A(f) : A[M_1, f^*F] \longrightarrow A[M_2, F]$,

$A(f)(\alpha)(h) = \mathcal{L}(fh)$, where $\mathcal{L} \in A_m[M_1, f^*F]$ and $h \in C^\infty_m(M_2, F)$.

If F is the foliation by points of the manifold M, $A[M,F] = A[M]$ - the bundle of A-points of M .

Example 3. Transverse natural bundles.

Let Fol_q be the category of smooth manifolds foliated by smooth codimension q foliations with smooth, foliation preserving transverse mapping to the foliation.

Definition. A covariant functor N on the category Fol_q into the category of locally trivial fibre bundles and their fibre mappings is called a transverse natural bundle if the following conditions are satisfied: i/for any foliated manifold (M,F), $N(M,F)$ is a fibre bundle over the manifold M ;

ii/if $f: (M_o,F_o) \longrightarrow (M_1,F_1)$ is a transverse mapping

such that $f^*F_1 = F_o$, then $N(f) : N(M_o,F_o) \longrightarrow N(M_1,F_1)$ covers f and maps the fibre $N(M_o,F_o)_x$ over x diffeomorphically onto the fibre $N(M_1,F_1)_{f(x)}$ over $f(x)$;

iii) N is a regular functor i.e. if $f : U \times M_o \longrightarrow M_1$ is a differentiable mapping, U an open subset of R^k , such that for any point t of the set U , the mapping $f_t : (M_o,F_o) \rightarrow (M_1,F_1)$ $f_t(x) = f(t,x)$ is a transverse mapping to the foliation and

$f_t^*F_1 = F_o$, then the mapping $U \times N(M_o,F_o) \ni (t,y) \longmapsto N(f_t)(y) \in N(M_1,F_1)$ is of class C^∞ .

Properties: 1/ For any morphism $f : (M_o,F_o) \rightarrow (M_1,F_1)$ the fibre bundle $f^*N(M_1,F_1)$ is isomorphic to $N(M_o,F_o)$.

2/ Let f,g be two morphisms of (M_o,F_o) into (M_1,F_1) such that $f(m) = g(m)$. Let (U,φ) be an adapted chart at m and (V,ψ) an adapted chart at $f(m)$. Since the mappings f and g preserve the foliation, the mappings $\bar{f} = \psi f \varphi^{-1} : R^{n-q} \times R^q \rightarrow R^{m-q} \times R^q$,

$\bar{g} = \psi g \varphi^{-1} : R^{n-q}xR^q \longrightarrow R^{m-q}xR^q$, where dim $M_o = n$, dim $M_1 = m$, are of the form $f(y,x) = (f_1(y,x), f_2(x))$, $g(y,x) = (g_1(y,x), g_2(x))$, where y denotes the first $n-q$ coordinates, x the last q coordinates, $f_1, g_1 : R^n \longrightarrow R^{m-q}$ and $f_2, g_2 : R^n \longrightarrow R^q$. If the germs of the mappings f_2 and g_2 at $\varphi^{-1}(m)$ are equal, then the mappings $N(f)$ and $N(g)$ define the same mapping on the fibre $N(M_o, F_o)_m$.

Proof. The property 1 is obvious. One has only to show the second. Let (U, φ), (V, ψ) be two adapted charts such that $f_2 | \varphi(U) = g_2 | \varphi(U)$ and $\varphi(U) = D^{n-q}xD^q$, where D^k denotes the k-disc. Assume that $\varphi(m) = 0$ and $\psi(f(m)) = 0$. Then the following diagram is commutative.

$$\begin{tikzcd}
U \arrow[r, "f|U\,,\ g|U"] \arrow[d] & V \arrow[d, dash] \\
D^{n-q}xD^q \arrow[r, "\bar{f}, \bar{g}"] & D^{m-q}xD^q \\
D^q \arrow[u, "i_o"] \arrow[r, "f_2\,,\ g_2"] & D^q \arrow[u, "p"']
\end{tikzcd}$$

where $i_o : D^q \longrightarrow D^{n-q}xD^q$ is given by $i_o(x) = (0,x)$ and $p : D^{n-q}xD^q \longrightarrow D^q$ by $p(y,x) = y$.

Since N is a functor, it is sufficient to show that the mappings f, and g induce the same mapping in the fibre over 0 . But since $p\bar{f}i_o = f_2$ and $p\bar{g}i_o = g_2$, $N(f_2) = N(p)N(\bar{f})N(i_o) = N(g_2) = N(p)N(\bar{g})N(i_o)$. But $N(p)$ and $N(i_o)$ induce isomorphisms on the fibre, hence the mappings $N(\bar{g})$ and $N(\bar{f})$ are equal on the fibre over 0, which ends the proof.

Definition. A transverse normal bundle N is finite order r if for any two morphisms $f, g : (M_o, F_o) \longrightarrow (M_1, F_1)$ the integer r is the smallest one for which the following implication is true:

$$j^r_x f = j^r_x g \implies N(f)(y) = N(g)(y) \quad \text{for any point } y \text{ of}$$

the fibre $N(M_0, F_0)_x$.

Having this definition we can prove the following theorem.

Theorem 1. Let N be a transverse natural bundle. Then there exists an integer r and an L^r_q-space W such that N is isomorphic to the fibre bundle associated to the transverse r-frame bundle with the standard fibre W . The smallest such integer r is the order of the transverse natural bundle N .

Proof. The first case to consider is that of a foliated manifold (M,F) whose foliation F is given by a global submersion $f : M \to M_0$. In this case the transverse r-frame bundle $L^r(M,F)$ is isomorphic to $f^*L^r(M_0)$. Let $B(L^r(M_0),W)$ be an associated fibre bundle to the r-frame bundle $L^r(M_0)$ with the standard fibre W . Then $f^*B(L^r(M_0),W)$ is an associated fibre bundle to the transverse r-frame bundle $L^r(M,F)$ with the standard fibre W . In what follows we shall denote $B(L^r(M_0),W)$ by $B(M_0,W)$ and $f^*B(L^r(M_0),W)$ by $B(M,F;W)$. The isomorphism from Palais-Terg's Theorem (cf. [6]) we shall denote by $B(M_0)$. Thus the following diagram is commutative:

$$\begin{tikzcd}
N(M,F) \arrow[r, "a"] \arrow[dr, "B(M,F)"'] & f^*N_0(M_0) \arrow[r, "\bar{f}"] \arrow[d, "B(M_0)^*"] & N_0(M_0) \arrow[d, "B(M_0)"] \\
 & f^*B(M_0) \arrow[r, "f"] & B(M_0,W)
\end{tikzcd}$$

where $B(M,F) = B(M_0)^* \circ a$, a is the isomorphism from Property 1 and N_0 is the natural bundle on the category of q manifolds obtained from N by foliating q manifolds by points. To this natural bundle we have applied Palais-Terng's Theorem. The mapping $B(M,F)$ is an isomorphism of fibre bundles since both mappings $B(M_0)^*$ and a are.

To complete this special case we have to show that these isomorphisms are functorial. Let F_1 be a foliation given by a global submersion $f_1 : M_1 \longrightarrow V_1$ and F_2 be a foliation given by a global submersion $f_2 : M_2 \longrightarrow V_2$ such that the diagram

$$\begin{tikzcd}
M_1 \arrow[r, "f"] \arrow[d, "f_1"'] & M_2 \arrow[d, "f_2"] \\
V_1 \arrow[r, "\bar{f}"] & V_2
\end{tikzcd}$$

is commutative for some smooth mapping $\bar{f}$. We have to show that the following diagram is commutative.

$$\begin{tikzcd}
N(M_1,F_1) \arrow[r, "N(f)"] \arrow[d, "B(M_1,F_1)"'] & N(M_2,F_2) \arrow[d, "B(M_2,F_2)"] \\
B(M_1,F_1;W) \arrow[r, "B(f)"] & B(M_2,F_2;W)
\end{tikzcd}$$

It is so, because the following diagram is commutative:

$$\begin{tikzcd}
N(M_1,F_1) \arrow[rr] \arrow[drrr, "B(M_1,F_1)"] \arrow[dd, "N(f)"'] & & f_1^*N_o(V_1) \arrow[rr] \arrow[dd, "N_o^*(f)"'] & & N_o(V_1) \arrow[dr, "B(V_1)"] \arrow[dd, "N_o(f)"'] & \\
 & & & B(M_1,F_1;W) \arrow[rr] \arrow[dd, "B(f)^*=B(f)"] & & B(V_1;W) \arrow[dd, "B(\bar{f})"] \\
N(M_2,F_2) \arrow[rr] \arrow[drrr, "B(M_2,F_2)"'] & & f_2^*N_o(V_2) \arrow[rr] \arrow[dr] & & N_o(V_2) \arrow[dr, "B(V_2)"] & \\
 & & & B(M_2,F_2;W) \arrow[rr] & & B(V_2;W)
\end{tikzcd}$$

The general case can be proved in the following way. Let (M,F) be a foliated manifold. Then there exists a covering $\mathcal{U} = \{U_\alpha\}_A$ and global submersion $f_\alpha : U_\alpha \longrightarrow V$ defining the foliation

$F \mid U_\alpha$. Since $N(U_\alpha, F|U_\alpha) \cong N(M,F)|U_\alpha$, the bundle $N(M,F)$ is isomorphic to $\coprod_{\alpha\in A} N(U_\alpha, F \mid U_\alpha)/\sim$, where the relation $\sim$ is induced by the natural identification of points in various U_α. For each α we have the isomorphism $B(U_\alpha, F|U_\alpha)$, $B(U_\alpha, F|U_\alpha)$: : $N(U_\alpha, F|U_\alpha) \longrightarrow B(U_\alpha, F|U_\alpha;W)$, which agrees with the relation $\sim$ and the corresponding relation for the bundles $B(U_\alpha, F|U_\alpha;W)$, $\alpha\in A$. Thus the bundle $\coprod_{\alpha\in A} N(U_\alpha, F|U_\alpha)/\sim$ is isomorphic to $\coprod_{\alpha\in A} B(U_\alpha, F|U_\alpha;W)/\sim$. This isomorphism induced by $B(U_\alpha, F|U_\alpha)$ we shall denote by $B_{\mathcal{U}}(M,F)$. Since the bundle $\coprod_{\alpha\in A} B(U_\alpha, F|U_\alpha;W)/\sim$ is isomorphic to the associated fibre bundle to the transverse r-frame bundle of (M,F) with the standard fibre W, $B(M,F;W)$, for each such covering $\mathcal{U} = \{U_\alpha\}$ we have defined an isomorphism of $N(M,F)$ onto $B(M,F;W)$ denoted by $B_{\mathcal{U}}(M,F)$. We have to show that this isomorphism does not depend on the covering $\mathcal{U}$. Let $\mathcal{U}$ and $\mathcal{V}$ be two such coverings and $\mathcal{W}$ be a covering which is finer then $\mathcal{U}$ and $\mathcal{V}$. Then the following diagram is commutative.

$$\begin{tikzcd}
N(M,F) \arrow[r] \arrow[dr] & \coprod N(U_i,F|U_i)/\sim \arrow[r] & \coprod B(U_i,F|U_i;W)/\sim \arrow[r] & B(M,F;W) \\
& \coprod N(W_j,F|W_j)/\sim \arrow[r] \arrow[u] & \coprod B(W_j,F|W_j;W)/\sim \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where the horizontal arrows have been defined previously, and the two vertical arrows are naturally defined. Thus $B_{\mathcal{U}}(M,F) = B_{\mathcal{W}}(M,F)$. Therefore these isomorphisms are independent of the choice of a covering $\mathcal{U}$, and we denote the isomorphism thus obtained by $B(M,F)$.

The only remaining thing to prove is to show that $B(M,F)$ define an isomorphism of the functors N and $B(\cdot;W)$.

Let $f:(M_1,F_1) \longrightarrow (M_2,F_2)$ be a morphism. We can choose a covering $\mathcal{U} = \{U_i\}_I$ of M_1 with submersions $g_i : U_i \longrightarrow V_i$ defining the foliation F_1, and a covering $\mathcal{W} = \{W_j\}_J$ of M_2, with submersions $h_j: W_j \longrightarrow Z_j$ defining the foliation F_2 such that for any $i\in I$ there exists $j\in J$ with the property: $f(U_i)\subset W_j$. Let us denote $f|U_i : U_i \longrightarrow W_j$ by f_i. Then f_i induces a mapping $\bar{f}_i: V_i \longrightarrow Z_j$. Because N and $B(\cdot,W)$ are functors, the diagrams

$$\begin{tikzcd}
N(M_1,F_1) \arrow[d, "N(f)"'] & \coprod N(U_i,F_1U_i)/\sim \arrow[l] \arrow[d, "\coprod N(f_i)"] & \coprod B(U_i,F_1\ U_i;W)/\sim \arrow[d, "\coprod B(f_i)"'] & B(M_1,F_1;W) \arrow[l] \arrow[d, "B(f)"'] \\
N(M_2,F_2) & \coprod N(W_j,F_2W_j)/\sim \arrow[l] & \coprod B(W_j,F_2\ W_j;W)/\sim & B(M_2,F_2;W) \arrow[l]
\end{tikzcd}$$

are commutative.

Because the foliations $F_1|U_i$ and $F_2|W_j$ are globally defined by submersions, it follows from the first part of the proof that the diagram

$$\begin{tikzcd}
\coprod N(U_i,F_1\ U_i)/\sim \arrow[r] \arrow[d, "\coprod N(f_i)"'] & \coprod B(U_i,F_1\ U_i;W)/\sim \arrow[d, "\coprod B(f_i)"] \\
\coprod N(W_j,F_2\ W_j)/\sim \arrow[r] & \coprod B(W_j,F_2\ W_j;W)/\sim
\end{tikzcd}$$

commutes, which effectively ends the proof.

From our point of view, the bundles considered in Examples 1,2,3 are nothing else but the inverse images of the (p,r)-velocities, A-bundles and natural bundles, respectively, on the manifold M_F via the mappings f_i. They glue together because diffeomorphisms can be lifted to these bundles. Thus for these structures and geometrical objects connected with them we have the following "dictionary". On the left hand side there are transverse objects on the foliated manifold (M,F), and on the right hand side there are corresponding objects on the manifold M_F .

normal bundle of order r	tangent bundle of order r
bundle of transverse (p,r)-velocities	bundle of (p,r)-velocities
bundle of transverse A-points	bundle of A-points
transverse natural bundle	natural bundle
foliated (p,r)-tensor	(p,r)-tensor
transversely projectible G-structure	G-structure
basic r-form	r-form
basic connection	connection

One can lift foliated tensors and basic connections to the transverse bundles mentioned above by repeating the constructions for the corresponding bundles on manifolds with only minor changes. But this process is very tedious (cf.[7]). Using the correspondence explained in the "dictionary" we can prove it in the following way. Let t be a transverse object on (M,F). If we can lift such an object to a transverse bundle B of the type considered, the corresponding object on the manifold M_F can be lifted to the corresponding bundle on the manifold M_F, and this lift is left invariant by the lifts of the transformations g_{ij}, since any transverse object projected onto M_F is left invariant by g_{ij}. Inversely, any object on M_F invariant by g_{ij} can be lifted to (M,F). Therefore we have to check the following. Let t and t' be two objects of a given type on manifolds N and N', respectively, and f be a diffeomorphism of N onto N', such that $t = f^{*}t'$. Let L be the lifting considered. Then $L(t) = L(f)^{*}L(t')$.

First, we shall use this procedure to lift foliated tensor fields. Let t be a foliated tensor field of type (p,s) on the foliated manifold (M,F). Then t defines a tensor field $\tilde{t}$ of type (p,s) on the manifold M_F such that $t|U_i = f_i^{*}\tilde{t}$ and $g_{ij}^{*}\,\tilde{t}|f_j(U_i \cap U_j) = \tilde{t}|f_i(U_i \cap U_j)$.
We shall lift foliated tensor fields to the following transverse bundles.

i/ Normal bundle of order r.
The normal bundle $N^r(M,F)$ of order r admits a foliation F^r modelled on the manifold $T^r(M_F)$ with transformations $T^r(g_{ij})$. To these bundles we shall be able to lift any foliated tensor field; thus we have to check

$$(f^{*})^{(\lambda)} = N^r(f)^{*}t^{(\lambda)} \quad \text{for } \lambda = 0,1,\dots,r .$$

ii/ Bundle of transverse (p,r)-velocities.
The bundle of transverse (p,r)-velocities $N^{p,r}(M,F)$ on the foliated manifold (M,F) has a natural foliation F_p^r. The foliation F_p^r is modelled on $T^{p,r}(M_F)$ with $T_p^r(g_{ij})$ as transformations. Therefore to be able to lift foliated tensor fields we have to check that if $f^{*}t = t'$ then $T_p^r(f)^{*}t^{(\lambda)} = t'^{(\lambda)}$ for any $\lambda \in N(p,r)$.

iii/ Bundle of transverse A-points.
The bundle $A(M,F)$ of transverse A-points of the foliated manifold (M,F) admits a foliation F_A modelled on the manifold $A(M_F)$ with transformations $A(g_{ij})$. To these bundles we shall lift foliated

tensor fields of type (0,s) or (1,s); thus we have to check

$$(f^{*}t)^{(\lambda)} = A(f)^{*}t^{(\lambda)} \qquad \text{for } \lambda = 0,..,N \ .$$

iv/ Transverse natural bundles.
Let $N(M.F)$ be a transverse natural bundle on the foliated manifold (M,F). Then the manifold $N(M,F)$ admits a foliation F_N of the same dimension as F modelled on $N(M_F)$ with transformations $N(g_{ij})$. We shall be able to lift foliated tensors of type (1,s) but only if the functor N fulfils the following condition:

(o) The union of all open orbits of the action of L^r_q on the standard fibre W is dense in W.
The condition (o) allows to define the complete lift of tensor fields of type (1,s) to this natural bundle (cf. [1]).

Because the lifts for the first three types are defined multiplicatively, we have to check the equalities only for functions, 1-forms and vector fields. The case of natural bundles will be dealt separately.

a/ Let $h \in C^{\infty}(M)$, $f : M \to M$ be a local diffeomorphism. We have to show that

i/ $\qquad (hf)^{(\lambda)} = h^{(\lambda)}T^r(f) \qquad$ for $\lambda = 0,..,r$;

ii/ $\qquad (hf)^{(\lambda)} = h^{(\lambda)}T^r_p(f) \qquad$ for $\lambda \in N(p,r)$;

iii/ $\qquad (hf)^{(\lambda)} = h^{(\lambda)}A(f) \qquad$ for $\lambda = 0,..,N$.

The equalities follow directly from the definitions.

b/ Let X be a vector field on M, $f : M \to M$ be a local diffeomorphism.
We have to show that

i/ $\qquad T^r(f)_{*}X^{(\lambda)} = (f_{*}X)^{(\lambda)} \qquad$ for $\lambda = 0,..,r$;

ii/ $\qquad T^r_p(f)_{*}X^{(\lambda)} = (f_{*}X)^{(\lambda)} \qquad$ for $\lambda \in N(p,r)$;

iii/ $\qquad A(f)_{*}X^{(\lambda)} = (f_{*}X)^{(\lambda)} \qquad$ for $\lambda = 0,..,N$.

Since the proofs of these equalities are similar, we shall show it only for A-bundles.

$$\begin{aligned} A(f)^{*}w^{(\lambda)}(X^{(\gamma)}) &= w^{(\lambda)}(A(f)_{*}X^{(\gamma)}) \\ &= w^{(\lambda)}((f_{*}X)^{(\gamma)}) \\ &= \sum c^{\mu}_{\lambda\gamma}\ (w(f_{X}X))^{(\mu)} \\ &= \sum c^{\mu}_{\lambda\gamma}\ (f^{*}w(X))^{(\ )} \\ &= (f^{*}w)^{(\lambda)}(X^{(\gamma)}) \end{aligned}$$

for any vector field X, $\lambda, \gamma = 0,\dots,N$. Therefore $A(f)^{*}w^{(\lambda)} = (f^{*}w)^{(\lambda)}$.

Let t be a foliated tensor field on the manifold (M,F), and let $t^{(\mu)}$ be the lift to one of the considered bundles obtained in the above way. Then the tensor field $t^{(\mu)}$ has the following properties.

i/ $\quad \mathcal{L}^{k}_{X^{(\lambda)}}t^{(\mu)} = (\mathcal{L}^{k}_{X}t)^{(\mu+\lambda-r)} \qquad$ for $\lambda,\mu = 0,\dots,r$;

ii/ $\quad \mathcal{L}^{k}_{X\langle\lambda\rangle}t^{(\mu)} = (\mathcal{L}^{k}_{X}\ t)^{(\mu-\lambda)} \qquad$ for $\mu,\lambda \in N(p,r)$;

iii/ $\quad \mathcal{L}^{k}_{X^{(\lambda)}}t^{(\mu)} = \sum c^{\gamma}_{\lambda\mu}\ (\mathcal{L}^{k}_{X}\ t)^{(\gamma)} \quad$ for $\lambda,\mu,\gamma = 0,\dots,N$,

where X is a foliated vector field and $\mathcal{L}^{k}$ denotes the contraction. The equalities follow from the corresponding equalities for tensor fields on model manifolds. It is also clear that if a lifting of a tensor field fulfils suitable equality, it must be unique as the vector fields $X^{(\lambda)}$ span the whole tangent space. In this way we retrieved the results of lifting of foliated tensor fields to normal bundles of order r contained in [7] and proved the following.

Theorem 2. Let (M,F) be a foliated manifold.

Then for any $\lambda \in N(p,r)$, there exists the unique lifting

$L_{\lambda} : T^{\varepsilon}_{v}(M,F) \longrightarrow T^{\varepsilon}_{v}(N^{r}_{p}(M,F)\ ,\ F^{r}_{p})$, $\varepsilon = 0,1$ such that

$$\mathcal{L}^{k}_{X\langle\mu\rangle}L_{\lambda}t = L_{\lambda-\mu}(\mathcal{L}^{k}_{X}t) \quad \text{for any } \mu \in N(p,r)\ ,$$

and for any $\lambda = 0,\dots,N$, there exists the unique lifting

$$L_\lambda : T_V^\epsilon(M,F) \longrightarrow T_V(A(M,F), F_A)) , \epsilon = 0,1 , \text{ such that}$$

$$\mathcal{L}^k_{X(\mu)} L_\lambda t = \sum c^{\,\nu}_{\lambda\mu} L_\nu t \quad \text{for any } \mu = 0,\dots,N .$$

Now, we shall deal with the fourth case. For transverse natural bundles the following theorem is true.

Theorem 3. Let (M,F) be a foliated manifold, N a transverse natural bundle fulfilling the condition (o). For any foliated tensor field t of type $(1,s)$, there exists the unique lift t^C to the total space of the bundle $N(M,F)$ such that

$$t^C(X_1^C,\dots,X_s^C) = (t(X_1,\dots,X_s))^C$$

for any foliated vector fields X_i.

Proof. Let X be a foliated vector field and X' be its representative. Let φ_t be the flow of X'. It preserves the foliation F. Thus we can define $N(\varphi_t)$, which in its turn is the flow of a vector field $\widetilde{X}'$ on the total space of $N(M,F)$, and preserves the foliation F_N. Thus we have defined a foliated vector field on $N(M,F)$ which does not depend on the choice of X' and is denoted by X^C. On an open set U on which the foliation F is defined by a submersion f, X^C is an inverse image by $N(f)$ of the complete lift X_0^C of the corresponding vector field X_0. Thus X^C can be obtained as the inverse image of the complete lift of a vector field on the model manifold M_F, as directly from the definition $(f_*X)^C = N(f)_*X^C$, for any vector field X and any local diffeomorphism f.

Let t be a tensor field of type $(1,s)$ on the manifold N and f be a local diffeomorphism of N. Then

$$\begin{aligned} N(f)^* t^C(X_1^C,\dots,X_s^C) &= N(f^{-1})_* t^C(N(f)_*X_1^C,\dots,N(f)_*X_s^C) \\ &= N(f^{-1})_* t^C((f_*X_1)^C,\dots,(f_*X_s)^C) \\ &= N(f^{-1})_*(t(f_*(X_1),\dots,f_*(X_s)))^C \end{aligned}$$

$$= (f^{-1}{}_{*}t(f_{*}X_1,\dots,f_{*}X_s))^C$$
$$= (f^{*}t(X_1,\dots,X_s))^C$$
$$= (f^{*}t)^C(X_1^C,\dots,X_s^C)\ .$$

Thus because of the condition (o), $N(f)^{*}t^C = (f^{*}t)^C$. Therefore, each foliated tensor field t of type $(1,s)$ can be lifted to the total space of the transverse natural bundle $N(M,F)$. We put $t^C \mid U_i = f_i^{*}\, t_o^C$, where t_o is the corresponding tensor field on $f_i(U_i)$. Then

$$\begin{aligned} t^C(X_1^C,\dots,X_s^C)_v &= f_i^{*}t_o^C(X_1^C,\dots,X_p^C) \\ &= (dN(f_i))^{-1}t_o^C(df_i(X_1)^C,\dots,df_i(X_s)^C) \\ &= (dN(f_i))^{-1}t_o^C(X_{1o}^C,\dots,X_{so}^C) \\ &= (d(N(f_i)))^{-1}(t_o(X_{1o},\dots,X_{so}))^C \\ &= ((df_i)^{-1}t_o(X_{1o},\dots,X_{so}))^C \\ &= (t(X_1,\dots,X_s))^C \quad , \end{aligned}$$

where X_o is the corresponding vector field on M_F to the foliated vector field X . This ends the proof of Theorem 3 .

To complete this short paper we shall prove that the lift of basic connections to normal bundles of order r , bundles of transverse (p,r)-velocities and transverse A-bundles exist. Because we shall apply the same method as for tensor fields, we have to check only the following. If ∇ is a connection on a manifold M , ∇' a connection on a manifold M' , and f is a local diffeomorphism of M into M' such that $f_{*}\nabla_X Y = \nabla'_{f_{*}X}f_{*}Y$ for any two vector fields, then

i/ $T^r f_{*}\ \widetilde{\nabla}_{X^{(\lambda)}}Y^{(\mu)} = \widetilde{\nabla}'_{T^r f_{*}X^{(\lambda)}}T^r f_{*}Y^{(\mu)}$ for $\lambda, \mu = 0,\dots,r$;

ii/ $T_p^r f_{*}\ \widetilde{\nabla}_{X^{(\lambda)}}Y^{(\mu)} = \widetilde{\nabla}'_{T_p^r f_{*}X^{(\lambda)}}T_p^r f_{*}Y^{(\mu)}$ for $\lambda, \mu \in N(p,r)$;

iii/ $$A(f)_* \tilde{\nabla}_{X^{(\lambda)}} Y^{(\mu)} = \tilde{\nabla}'_{A(f)_* X^{(\lambda)}} A(f)_* Y^{(\mu)}$$
for $\lambda, \mu = 0,\ldots,N$.

where X, Y are vector fields, and $\tilde{\nabla}$ is the lift of the connection ∇ . The proof of these cases are similar. We shall check only the third one.

$$\begin{aligned}\tilde{\nabla}'_{A(f)_* X^{(\lambda)}} A(f)_* Y^{(\mu)} &= \tilde{\nabla}'_{(f_* X)^{(\lambda)}} (f_* Y)^{(\mu)} \\ &= \sum c^{\nu}_{\lambda\mu} (\nabla'_{f_* X} f_* Y)^{(\nu)} \\ &= \sum c^{\nu}_{\lambda\mu} (f_*(\nabla_X Y))^{(\nu)} \\ &= \sum c^{\nu}_{\lambda\mu} A(f)_*(\nabla_X Y)^{(\nu)} \\ &= A(f)_*(\sum c^{\nu}_{\lambda\mu} (\nabla_X Y)^{(\nu)}) \\ &= A(f)_*(\tilde{\nabla}_{X^{(\lambda)}} Y^{(\mu)}) \ .\end{aligned}$$

Theorem 4. Let (M,F) be a foliated manifold. Let ∇ be a basic connection. Let $B(M,F)$ be a transverse A-bundle. Then there exists a connection $\tilde{\nabla}$, basic for the foliation F_A of the total space of the bundle $B(M,F)$ such that

$$\tilde{\nabla}_{X^{(\mu)}} Y^{(\lambda)} = \sum c^{\nu}_{\mu\lambda} (\nabla_X Y)^{(\nu)} \ .$$

for any foliated vector fields X, Y on the manifold (M,F) and $\lambda, \mu = 0,\ldots,N$.

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INSTYTUT MATEMATYKI
UNIWERSYTET JAGIELLOŃSKI
UL. REYMONTA 4
30-059 KRAKÓW