Klaus Gürlebeck; Wolfgang Sprössig

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A GENERALIZED LEIBNIZ RULE AND FOUNDATION OF A DISCRETE QUATERNIONIC ANALYSIS

K. Gürlebeck/W. Sprössig

0. INTRODUCTION

This paper presents some new results in the real quaternionic analysis. As well known, the Leibniz rule of the classical complex function theory is given by the identity

$$\frac{\partial}{\partial z}(f\cdot g) = (\frac{\partial}{\partial z} f)g + f(\frac{\partial g}{\partial z}) \qquad (0.1)$$

where f,g are complex-valued functions defined over the field of complex numbers and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. Especially, it follows that the product of analytic functions is also an analytic function. The last assertion remains not valid for the case of quaternionic-valued functions. Therefore formula (1.1) cannot be transferred to higher dimensions in this simple way.

Let e_i , $i = 0,1,2,3$ be the quaternionic units which fulfil the following conditions

$$\begin{aligned} &e_o e_i = e_i e_o = e_i \;, && i = 0,1,2,3 \\ &e_i^2 = -e_o \;, && i = 1,2,3 \\ &e_i e_j + e_j e_i = 0 && i,j = 1,2,3 \;;\; i \neq j \;. \end{aligned} \qquad (0.2)$$

Further, given the real-valued functions f_i, g_i , $i = 0,1,2,3$ and the quaternionic-valued functions

$$f = \sum_{i=0}^{3} e_i f_i \quad \text{and} \quad g = \sum_{i=0}^{3} e_i g_i$$. Then we put $f_o = \operatorname{Re} f$ and $\hat{f} = \operatorname{Im} f = \sum_{i=1}^{3} e_i f_i$.

Introducing the differentiations by

This paper is in final form and no version of it will be submitted for publication elsewhere.

$$\bar{\partial}_l f = \frac{1}{2}\left(\frac{\partial f}{\partial x_0} e_0 + \sum_{i=1}^{3} e_i \frac{\partial f}{\partial x_i}\right) \text{ and } \bar{\partial}_r f = \frac{1}{2}\left(\frac{\partial f}{\partial x_0} e_0 + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} e_i\right)$$

and setting

$$D = e_0 dx_1 \wedge dx_2 \wedge dx_3 - e_1 dx_0 \wedge dx_2 \wedge dx_3 - e_2 dx_0 \wedge dx_3 \wedge dx_1 - e_3 dx_0 \wedge dx_1 \wedge dx_2$$

$$v = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$$

A. Sudbery [9] obtained the formula

$$d\left[g\ D\ f\right] = \left\{\left[\bar{\partial}_r g\right] f + g\left[\partial_l f\right]\right\} v \qquad (0.3)$$

Let now $\nabla = \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i}$ be the generalized Nabla-operator. Then we immediately obtain from a result in [8] the identity

$$\sum_{i=0}^{3} \frac{\partial}{\partial x_i}\left[\bar{f}\ e_i g\right] = \overline{\left[\overline{\nabla} f\right]} g + \bar{f}\left[\nabla g\right] \qquad (0.4)$$

where $\quad \bar{a} = a_0 e_0 - \sum_{i=1}^{3} a_i e_i$.

In Section 1 we deduce a generalized Leibniz rule for quaternionic-valued functions, whose left side issimilar to this of the relation (0.1).

In Section 2 of our paper we present a model of a discrete functiontheory of quaternions. Using this results we obtained a new approach to the foundation of numerical methods for solving partial differential equations. The definition of a discrete analytic function in the plane was firstly introduced by J. Ferrand in 1944 (see [2]). A complex-valued function f defined on the lattice $Z^+ \times Z^+$ is called discrete analytic, if it satisfies the condition

$$f(m,n) + i\ f(m+1,n) + i^2 f(m+1,n+1) + i^3 f(m,n+1) = 0 \quad .$$

Essential properties of such functions were obtained in papers of R. J. Duffin [1], S. Hayabara [6] and D. Zeilberger [10]. In the first part of the second section we prove the existence of the fundamental solution of the discrete Laplacian and point out a way of its approximative calculation. After this we consider discrete analogues to the operators from [5] by help of the fundamental solution and we investigate their algebraic properties. Most of them coincide with the corresponding ones of the continuous case. Therefore the basic methods of hypercomplex function theory can be also used in the discrete theory. Finally, we deduce formulas of representation of solutions of the discrete Laplace equation and describe the orthogonal complement of the space of discrete-analytic functions in L_2.

1. A LEIBNIZ RULE FOR QUATERNIONIC FUNCTIONS

Let $f = \sum_{i=0}^{3} f_i e_i$, $g = \sum_{i=0}^{3} g_i e_i$ quaternionic functions. The product between both functions is determined by

$$f\,g = \sum_{i=0}^{3}\sum_{j=0}^{3} f_i g_j e_i e_j \tag{1.1}$$

Obviously, in general, $fg \neq gf$.
Using the properties (0.2) the product (1.1) can be expressed by

$$\begin{aligned} fg = (f_0 g_0 - \sum_{i=1}^{3} f_i g_i) e_0 + \left[(f_2g_3-f_3g_2) + f_0g_1 + f_1g_0\right] e_1 + \\ + \left[(f_3g_1-f_1g_3) + f_0g_2+g_0f_2\right]e_2 + \left[(f_1g_2-f_2g_1)+f_0g_3+g_0f_3\right] e_3 \end{aligned} \tag{1.2}$$

Let $\mathbb{H}$ be the skew-field of quaternions. Then the set of real-differentiable quaternionic functions will be denoted by $C^1_{\mathbb{H}}$. Furthermore let ∂_i , $i = 0,1,2,3$, be the partial derivatives $\frac{\partial}{\partial x_i}$, $i = 0,1,2,3$.

The classical Cauchy-Riemann operator is generalized by $\nabla = \sum_{i=0}^{3} \partial_i e_i$ while $\overline{\nabla} = \partial_0 e_0 - \sum_{i=1}^{3} \partial_i e_i$ denotes the adjoint of ∇. Notice that $\nabla\overline{\nabla} = \overline{\nabla}\nabla = \Delta$, Δ being the Laplacian in $\mathbb{R}^4$.
We now state our product rule as follows

THEOREM 1
Let $f,g \in C^1_{\mathbb{H}}$. Then holds

$$\nabla(fg) = (\nabla f)g - \bar{f}(\overline{\nabla} g) + 2\,\mathrm{Re}(f\nabla)g \tag{1.3}$$

Proof.
Using the product (1.1) it follows

$$\nabla(fg) = \nabla \sum_{i=0}^{3}\sum_{j=0}^{3} f_i g_j e_i e_j = \sum_{k=0}^{3} \partial_k e_k \left(\sum_{i=0}^{3}\sum_{j=0}^{3} f_i g_j e_i e_j\right) =$$

$$= \sum_{i,j,k=0}^{3} \partial_k (f_i g_j) e_k e_i e_j = \sum_{i,j,k=0}^{3} \left[(\partial_k f_i) g_j + f_i (\partial_k g_j)\right] e_k e_i e_j$$

Setting

$$S_1 = \sum_{i,j,k=0}^{3} (\partial_k f_i) g_j e_k e_i e_j \quad \text{and} \quad S_2 = \sum_{i,j,k=0}^{3} f_i (\partial_k g_j) e_k e_i e_j$$

one obtains for S_1

$$S_1 = \sum_{i=0}^{3} \sum_{k=0}^{3} \partial_k f_i e_k e_i \sum_{j=0}^{3} g_j e_j = (\nabla f) g \quad .$$

Following algebraic calculations are necessary for the treatment of the term S_2 .

$$S_2 = \sum_{j=0}^{3} \sum_{k=1}^{3} \sum_{i=1}^{3} f_i (\partial_k g_j) e_k e_i e_j + \sum_{j=0}^{3} f_o e_o \partial_o g_j e_j +$$

$$+ \sum_{j=0}^{3} \sum_{i=1}^{3} f_i (\partial_o g_j) e_i e_j + \sum_{j=0}^{3} \sum_{k=1}^{3} f_o (\partial_k g_j) e_k e_j =$$

$$= -\left[\sum_{k=1}^{3} \sum_{i=1}^{3} f_i e_i \partial_k e_k \right] g - 2 \sum_{i=1}^{3} f_i \partial_i g + f_o \partial_o g +$$

$$+ \sum_{i=1}^{3} f_i e_i \partial_o g + \sum_{k=1}^{3} f_o \partial_k e_k g = \left[\sum_{k=1}^{3} \bar{f} \partial_k e_k \right] g -$$

$$- 2 \sum_{i=1}^{3} f_i \partial_i g + f_o \partial_o g + \sum_{i=1}^{3} f_i e_i \partial_o g = - \bar{f} (\bar{\nabla} g) +$$

$$+ \bar{f} \partial_o g - 2 \sum_{i=1}^{3} f_i \partial_i g + f \partial_o g = - \bar{f} (\bar{\nabla} g) + 2 \operatorname{Re}(f \nabla) g$$

Finally the sum $S_1 + S_2$ yields the wanted formula.

<u>Special case</u>

If $\partial_o f = \partial_o g = 0$, then (1.3) leads to the formula

$$\hat{\nabla}(fg) = (\hat{\nabla} f) g + \bar{f} (\hat{\nabla} g) - 2 \left[\operatorname{Re}(f \hat{\nabla}) \right] g \qquad (1.4)$$

where $\hat{\nabla} = \sum_{i=1}^{3} \partial_i e_i$.

A connection between the operator $\hat{\nabla}$ and the basic terms of the vector analysis is given by

$$1.\quad \hat{\nabla} f_o e_o \underset{def}{=} (\text{grad } f_o) e_o$$

$$2.\quad \hat{\nabla} \hat{f} \underset{def}{=} (-\text{div } \hat{f}) e_o + \text{rot } \hat{f} \qquad (1.5)$$

where $\text{rot } \hat{f} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$, $\text{div } \hat{f} = \sum_{i=1}^{3} \partial_i f_i$.

<u>Corollary 1 (Product rules of the vector analysis)</u>
When $f = f_o e_o + \hat{f}$, $g = g_o e_o + \hat{g}$ and both functions belong to $C^1_{\mathbb{H}}$ then are valid the following relations.

(i) $\text{grad } f_o g_o = (\text{grad } f_o) g_o + f_o (\text{grad } g_o)$
(ii) $\text{div}(f_o \hat{g}) = (\text{grad } f_o) \hat{g} + f_o \text{ div } \hat{g}$
(iii) $\text{rot}(f_o \hat{g}) = \text{grad } f_o \times \hat{g} + f_o \text{ rot } \hat{g}$
(iv) $\text{div } \hat{f} \times \hat{g} = \hat{g} \cdot \text{rot } \hat{f} - \hat{f} \cdot \text{rot } \hat{g}$
(v) $\text{grad } \hat{f} \cdot \hat{g} = \hat{f} \times \text{rot } \hat{g} + \hat{g} \times \text{rot } \hat{f} + (\hat{f} \cdot \text{grad}) \hat{g} + (\hat{g} \cdot \text{grad}) \hat{f}$
(vi) $\text{rot } \hat{f} \times \hat{g} = \hat{f} \text{ div } \hat{g} - \hat{g} \text{ div } \hat{f} + (\hat{g} \cdot \text{grad}) \hat{f} - (\hat{f} \cdot \text{grad}) \hat{g}$.

<u>Proof.</u>
First setting $f = f_o e_o$, $g = g_o e_o$. Using (1.4) it follows

$$\hat{\nabla}(f_o g_o e_o) = (\hat{\nabla} f_o e_o) g_o e_o + f_o e_o (\hat{\nabla} g_o e_o) + 0$$

Taking into consideration (1.5) we obtain (i). If we put $f = f_o e_o$, $g = \hat{g}$ it follows immediately

$$\hat{\nabla}(f_o \hat{g}) = (\hat{\nabla} f_o e_o) \hat{g} + f_o (\hat{\nabla} \hat{g}) + 0$$

Making use of the notation (1.5) then the definition of the quaternionic product yields

$$-(\text{div} f_o \hat{g}) e_o + \text{rot} f_o \hat{g} = -(\text{grad} f_o) \hat{g}\, e_o + \text{grad} f_o \times \text{rot} \hat{g} - (f_o \text{div} \hat{g}) e_o + f_o \text{rot} \hat{g} .$$

This proves the statements (ii) and (iii). Now we set $f = \hat{f}, g = \hat{g}$. Then formula (1.4) gives

$$\hat{\nabla}(\hat{f}\hat{g}) = (\hat{\nabla} \hat{f}) \hat{g} - \hat{f}(\hat{\nabla} \hat{g}) - \left(\sum_{i=1}^{3} f_i \partial_i \right) \hat{g} \qquad (1.6) .$$

Applying (1.5) and (1.2) the left side of (1.6) can be expressed by

$$\hat{\nabla}(-\hat{f}\cdot\hat{g}\,e_0+\hat{f}\times\hat{g}) = -\operatorname{div}(\hat{f}\times\hat{g})e_0 - \operatorname{grad}\hat{f}\cdot\hat{g} + \operatorname{rot}\hat{f}\times\hat{g}$$

while the right side of (1.6) can be transformed into

$$\left[(-\operatorname{div}\hat{f})e_0+\operatorname{rot}\hat{f}\right]\hat{g} - \hat{f}\left[(-\operatorname{div}\hat{g})e_0+\operatorname{rot}\hat{g}\right] - \Big(\sum_{i=1}^{3} f_i\partial_i\Big)\hat{g} = \qquad (1.7)$$

$$= -(\operatorname{rot}\hat{f})\cdot\hat{g}\,e_0 + \operatorname{rot}\hat{f}\times\hat{g} - (\operatorname{div}\hat{f})\hat{g} + (\operatorname{div}\hat{g})\hat{f} +$$

$$+ (\hat{f}\cdot\operatorname{rot}\hat{g})e_0 - \hat{f}\times\operatorname{rot}\hat{g} - \Big(\sum_{i=1}^{3} f_i\partial_i\Big)\hat{g}$$

Interchanging the functions f and g we get

$$-\operatorname{grad}(\hat{f}\cdot\hat{g})-\operatorname{rot}\hat{f}\times\hat{g} = (\operatorname{rot}\hat{g})\times\hat{f}-(\operatorname{div}\hat{g})\hat{f}+(\operatorname{div}\hat{f})\hat{g}-\hat{g}\times\operatorname{rot}\hat{f}-(\hat{g}\cdot\operatorname{grad})\hat{f} \quad (1.8).$$

Adding of (1.7) and (1.8) proves the relation (v) while subtracting of these identities shows the validity of (vi) and (iv).

Corollary 2 (Borel-Pompeiu formula in $\mathbb{R}^4$)
Let G be a bounded domain in $\mathbb{R}^4$, whose boundary is a piece-wise-smooth Liapunov surface Γ. Furthermore let f $C^1_{\mathbb{H}}(G)\cap C_{\mathbb{H}}(G\cup\Gamma)$. Then

$$\frac{1}{2\pi^2}\int_\Gamma \frac{\alpha\,\theta}{|x-y|^3}\,f\,d\Gamma - \frac{1}{2\pi^2}\int_G \frac{\theta(\nabla f)}{|x-y|^3}\,dG = \begin{cases} f(x), & x\in G \\ 0, & x\notin \bar{G}\end{cases} \qquad (1.9)$$

where $\alpha = \sum_{i=0}^{3}\alpha_i e_i$ is the unit vector of the outer normal on Γ at the point y

and $\theta = \sum_{i=0}^{3}\theta_i e_i$, $\theta_i = \frac{y_i-x_i}{|y-x|}$.

Proof.
Setting in Theorem 1 $E = \frac{1}{2\pi^2}\frac{1}{|x-y|^2}e_0$ it follows

$$\nabla(Ef) = (\nabla E)f - E(\bar{\nabla}f) + 2\operatorname{Re}(E\nabla)f \quad .$$

By integrating over the domain G we obtain the identity

$$\int_G \nabla(Ef)dG = \int_G (\nabla E)f\,dG - \int_G E(\bar{\nabla}f)dG + 2\operatorname{Re}\int_G E(\nabla f)dG \quad .$$

Applying Green's theorem it follows

$$\int_{\Gamma} \alpha\, Ef\, d\Gamma = \int_G \frac{\theta\, f}{|x-y|^3}\, dG - \int_G E(\bar{\nabla} f)\, dG + \frac{1}{2\pi^2} \int_G \frac{2}{|x-y|^2}\, \partial_0 f\, dG \quad (1.10)$$

It is easy to see that

$$- E(\bar{\nabla} f) + \frac{1}{2\pi^2} \frac{2}{|x-y|^2}\, \partial_0 f = E(\nabla f) .$$

From (1.10) we deduce the relation

$$\frac{1}{2\pi^2} \int_{\Gamma} \alpha\, E\, f\, d\Gamma = \frac{1}{2\pi^2} \int_G \frac{\theta\, f}{|x-y|^2}\, dG + \frac{1}{2\pi^2} \int_G \frac{1}{|x-y|^2}\, \nabla f\, dG .$$

The differentiation by the operator $\nabla = \sum_{i=0}^{3} \partial_i e_i$ completes the proof.

Remark 1.

Let $G \subset \mathbb{R}^3$ be a bounded domain whose boundary is a piecewise-smooth Liapunov surface Γ. Setting

$E = \frac{1}{4\pi} \frac{1}{|x-y|}$, $\alpha = \sum_{i=1}^{3} \alpha_i e_i$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ the unit vector of the outer normal on Γ and $\theta = \sum_{i=1}^{3} \frac{x_i - y_i}{|x-y|} e_i$.

Then

$$\frac{1}{4\pi} \int_{\Gamma} \frac{\alpha\, \theta}{|x-y|^2}\, f\, d\Gamma + \frac{1}{4\pi} \int_G \frac{\theta(\nabla f)}{|x-y|^2}\, dG = \begin{cases} f(x) , & x \in G \\ 0 , & x \notin \bar{G} \end{cases} \quad (1.11)$$

Remark 2.

Introducing the operators

$$Ff = \int_{\Gamma} \alpha\, (\nabla E)\, f\, d\Gamma \quad \text{and} \quad Tf = \int_G (\nabla E) f\, dG .$$

Then we find

$$Ff + T\nabla f = \begin{cases} f , & \text{in } G \\ 0 , & \text{in co } \bar{G} \end{cases} .$$

The following theorem is very useful for describing the image of the operator T.

Corollary 3

Let G be a bounded domain in $\mathbb{R}^4$ or $\mathbb{R}^3$, $\Gamma = \partial G$. Moreover we assume that the boundary Γ split into the subsets Γ_1 and Γ_2 where $\Gamma_1 \cap \Gamma_2 = \phi$ and $\Gamma_1 \cup \Gamma_2 = \Gamma$. If we put $f_0 = 0$ on Γ_1, $g = 0$ on Γ_2 and $g \in \ker \nabla$, we have the identities

$$T[(\nabla f_0)g] = f_0 g$$

and

$$\gamma_0 T(\nabla f_0)g = 0$$

where γ_0 is the trace operator onto the boundary Γ.

Proof.

From (1.4) we see

$$\nabla(f_0 g) = (\nabla f_0)g + f_0(\nabla g) . \tag{1.12}$$

Since $\nabla g = 0$, the identity (1.12) yields by application of the operator T

$$(I - F)f_0 g = T\left[(\nabla f_0)g\right]$$

and thereby

$$f_0 g = T\left[(\nabla f_0)g\right] .$$

Corollary 4.

Let $f_0 g$, $g \in \ker \nabla$. Then we have either $f_0 =$ const. or $g = 0$.

Proof.

From Theorem 1 it immediately follows

$$0 = \nabla(f_0 g) = (\nabla f_0)g$$

and the proof is complete.

Remark 3.

It is worth noticing that the above proved Leibniz rule can be also formulated in the difference calculus. To do this let us fix the following notations

$$\partial_i^h f = \frac{f(x+e_i h)-f(x)}{h} , \quad h \in \mathbb{R} , \quad D_h = \sum_{i=0}^{3} \partial_i^h e_i ,$$

$$\mathbf{D}_h F = (\partial_0^h e_0, \partial_1^h e_1, \partial_2^h e_2, \partial_3^h e_3) \cdot F , \quad \overline{\mathbf{D}}_h G = (\partial_0^h \overline{e}_0, \partial_1^h \overline{e}_1, \partial_2^h \overline{e}_2, \partial_3^h \overline{e}_3) \cdot G ,$$

$F = (f,f,f,f)$, $G = (g,g,g,g)$, $S_h g = (g(x+e_0 h),\ldots,g(x+e_3 h))$. The sign "." means the usual inner product in the space $\mathbb{H}^4$. Then

$$D_h(fg) = \mathbf{D}_h F \cdot S_h g - F\overline{\mathbf{D}}_h G + 2\left[\mathrm{Re}(F\,\mathbf{D}_h)\right]\cdot G \,. \qquad (1.13)$$

2. FOUNDATION OF A DISCRETE QUATERNIONIC FUNCTION THEORY

2.1 Fundamental solution of the discrete Laplacian $-\Delta_h$

Definition 1

Let $\mathbb{R}_h^3 = \{(ih,jh,kh);\ i,j,k \in \mathbb{Z}\ ,\ h\in\mathbb{R},\ h = \text{const}\},\ G\subset\mathbb{R}_h^3$

$$Q_l = ([-l,l]\times[-l,l]\times[-l,l]) \cap \mathbb{R}_h^3 \quad,\quad \mathring{Q}_l = \text{int}\ Q_l \quad,$$

$l = N\cdot h,\quad N\in\mathbb{N}\quad,\quad \gamma_0$ the trace operator

$$(-\Delta_h u)(x) = A(x)u(x) - \sum_{\xi\in U(x)\setminus\{x\}} B(x,\xi)u(\xi)\ ,\qquad x\in\mathbb{R}_h^3\ ,$$

$$A = \frac{6}{h^2}\quad,\ B(x,x\pm h) = \frac{1}{h^2}\quad,$$

$$\langle f,g\rangle_{L_2(G)} = \sum_{x\in G} f(x)g(x)h\quad,$$

$$L_2(G) = \{ f :\ \langle f,f\rangle_{L_2(G)}$$

$$K_h(x) = \begin{cases} 1/h^3 & x=0\\ 0 & x\neq 0\end{cases}\,.$$

Applying well-known theorems of the theory of finite difference methods we get the following statements.

Lemma 1 [7].

The boundary value problem

$$\begin{aligned}(-\Delta_h u)(x) &= K_h(x) &&\text{in } \mathring{Q}_l\\ \gamma_0 u &= 0 &&\text{on } \partial Q_l\end{aligned}\qquad (2.1)$$

has the unique solution $u_{l,h}$ and it holds the unequality

$$u_{l,h}(x) \geqslant 0 \qquad \forall\ x\in Q_l \qquad (2.2)$$

Proof. discrete maximum principle

Lemma 2.

Let be $L \geq l$. Assuming $u_{L,h}$, $u_{l,h}$ are solutions of (2.1) in Q_L, Q_l, respectively. If the functions $u_{L,h}$, $u_{l,h}$ in co Q_L, co Q_l, respectively, are continued by zero then is the unequality

$$u_{L,h}(x) \geq u_{l,h}(x) \qquad \forall\, x \in R_h^3 \tag{2.3}$$

valid.

Proof. $K_l = K_L$ in Q_l, $u_L \geq u_l$ on Q_l, maximum principle.

Lemma 3.

$$u_{l,h}(0) \geq u_{l,h}(x) \qquad \forall\, x \in Q_l \tag{2.4}$$

Proof.

To prove this unequality suppose, on the contrary, that exists a point $x_0 \in Q_l \setminus \{0\}$ with $u_{l,h}(x_0) \geq u_{l,h}(x)$ $\forall\, x \in Q_l \setminus \{x_0\}$. Then it follows the existence of $Q_{l'}$ contained in Q_l with $x_0 \in Q_{l'}$, $\{0\} \notin \mathring{Q}_{l'}$, $Q_{l'} \cap \partial Q_l \neq \phi$ and therefore $u_{l,h}$ satisfies

$$-\Delta_h u_{l,h}(x) = 0 \quad \text{in } Q_{l'}$$

$$u_{l,h}(x) \geq 0 \quad \text{on } \partial Q_{l'}$$

The assumption and the maximum principle yield

$u_{l,h} \equiv$ const in $\overline{Q}_{l'}$ and consequently $u_{l,h} \equiv 0$ in $Q_{l'}$. Thereby it proves the Lemma 3.

We now consider the limit $l \to \infty$ for the construction of a fundamental solution $E_h(x)$. Expanding the function $u_{l,h}$ in a series of eigenfunctions of the operator $-\Delta_h$ one can obtain estimates of $E_h(x)$.

Lemma 4.

The functions

$$\varphi_{ijk}^{(l,h)}(x_1,x_2,x_3) = \frac{1}{l^{3/2}} \cos \frac{2i+1}{2l}\pi x_1 \cos \frac{2j+1}{2l}\pi x_2 \cos \frac{2k+1}{2l}\pi x_3$$

$$i,j,k = 0,1,\dots,N-1 \tag{2.5}$$

are the normed eigenfunctions of the operator $-\Delta_h$ in Q_l and

$$\lambda_{ijk}^{(l,h)} = \frac{4}{h^2}\left[\sin^2 \frac{2i+1}{4l}\pi h + \sin^2 \frac{2j+1}{4l}\pi h + \sin^2 \frac{2k+1}{4l}\pi h\right] \tag{2.6}$$

$i,j,k = 0,1,\dots,N-1$ are the corresponding eigenvalues.

The relations (2.5) and (2.6) lead to the following representation of $u_{1,h}$.

Lemma 5.

$$u_{1,h}(x_1,x_2,x_3) = \sum_{i,j,k=0}^{N-1} \frac{1}{1^{3/2}\,\lambda_{ijk}^{(1,h)}} \varphi_{ijk}^{(1,h)}(x_1,x_2,x_3) \text{ in } Q_1 \qquad (2.7)$$

Proof. Fourier series.

The following Lemma 6 is necessary to get an estimate for the function $u_{1,h}$.

Lemma 6.

Let $N = 1/h$ with a fixed number h. Then holds

$$\sum_{i,j,k=0}^{N-1} \frac{1}{(2i+1)^2+(2j+1)^2+(2k+1)^2} \leqslant C_1 1 + C_2 \qquad (2.8)$$

with constants C_1, C_2 independently of 1.

Proof. Let be

$$V_{ijk} = \left[(i-\tfrac{1}{2})h,(i+\tfrac{1}{2})h\right] \times \left[(j-\tfrac{1}{2})h,(j+\tfrac{1}{2})h\right] \times \left[(k-\tfrac{1}{2})h,(k+\tfrac{1}{2})h\right]$$

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \quad \theta \in V_{ijk}, \quad r_{ijk} = (i^2+j^2+k^2)^{1/2} h$$

then we obtain

$$\left| \int_{V_{ijk}} \frac{1}{r^2}\, dG - \frac{1}{r_{ijk}^2} h^3 \right| \leqslant \sup_{V_{ijk}} \frac{|x_{1,\theta}|+|x_{2,\theta}|+|x_{3,\theta}|}{r_\theta^4} h^4 \leqslant \sqrt{3}\, \frac{1}{r_\theta^3} h^4 \leqslant$$

$$\leqslant \sqrt{3}\, h^4 \frac{1}{(r_{ijk} - \frac{\sqrt{3}h}{2})^3} .$$

For any C and $r > (1 + \frac{3\sqrt{3}}{2} C)\frac{h}{C}$ we have $\frac{h}{(r-\frac{\sqrt{3}h}{2})^3} \leqslant \frac{C}{r^2}$ whence

$$\frac{1}{r_{ijk}^2} h^3 \leq \int_{V_{ijk}} \frac{1}{r^2} dG + \sqrt{3}\, C \frac{1}{r_{ijk}^2} h^3 .$$ That means for $C < 1/\sqrt{3}$

$$\frac{1}{r_{ijk}^2} h^3 \leq \frac{1}{1-\sqrt{3}\,C} \int_{V_{ijk}} \frac{1}{r^2} dG \quad . \tag{2.9}$$

If we choose, for instance, $C = 0.5$ then $r \geq 5\,h$ is sufficiently for (2.9). Now we estimate ($C = 0.5$)

$$\sum_{i,j,k=0}^{N-1} \frac{1}{(2i+1)^2+(2j+1)^2+(2k+1)^2} \leq \frac{1}{h} \sum_{i,j,k=1}^{N} \frac{1}{(i^2+j^2+k^2)h^2} h^3 =$$

$$= \frac{1}{h} \sum_{\substack{i,j,k \\ i^2+j^2+k^2 \leq 16}} \frac{1}{(i^2+j^2+k^2)h^2} h^3 + \frac{1}{h} \sum_{\substack{i,j,k \\ i^2+j^2+k^2 \geq 25}} \frac{1}{r_{ijk}^2} h^3 \leq$$

$$\leq C_2 + \frac{1}{4h} \sum_{Q_1 \setminus Q_{4h}} \frac{1}{r_{ijk}^2} h^3 \leq C_2 + \frac{1}{4h} \frac{1}{1-\sqrt{3}\,C} \sum_{ijk} \int_{V_{ijk}} \frac{1}{r^2} dG \leq$$

$$\leq C_2 + \frac{1}{4h} \frac{1}{(1-\sqrt{3}\,C)} \int_{K_{\sqrt{3}1}} r^{-2} dG = C_2 + \frac{1}{4h} \frac{\sqrt{3}}{1-\sqrt{3}\,C} 1 \tag{2.10}$$

<u>Lemma 7.</u> The sequence $\{u_{1,h}(x)\}_1$ is convergent $\forall x \in \mathbb{R}_h^3$.

<u>Proof.</u> From (2.3) and (2.4) it follows that it is sufficiently to show the boundedness of $\{u_{1,h}(0)\}$. We have

$$|u_{1,h}(0)| \leq \frac{h^2}{41^3} \sum_{i,j,k=0}^{N-1} \frac{1}{\sin^2 \frac{(2i+1)\pi h}{41} + \ldots + \sin^2 \frac{(2k+1)\pi h}{41}} \leq$$

$$\leq \frac{h^2}{4l^3} \sum_{i,j,k=0}^{N-1} \frac{1}{\frac{4}{\pi^2}\left(\frac{(2i+1)^2\pi^2h^2}{16l^2} + \dots + \frac{(2k+1)^2\pi^2h^2}{16l^2}\right)} \leq$$

$$\leq \frac{1}{l} \sum_{i,j,k=0}^{N-1} \frac{1}{(2i+1)^2+(2j+1)^2+(2k+1)^2} \leq \frac{1}{l}\left[c_2 + \frac{1}{4h} \frac{\sqrt{3}}{(1-\sqrt{3}c)} l \right] =$$

$$= \frac{c_2}{l} + \frac{1}{4h} \frac{\sqrt{3}}{1-\sqrt{3}\, c} \quad .$$

Definition 2. $E_h(x) = \lim_{l\to\infty} u_{l,h}(x) \quad , \quad x \in \mathbb{R}_h^3$.

Remark 4. In any bounded domain $G \subset \mathbb{R}_h^3$ $\{u_{l,h}(x)\}$ uniformely converges with respect to x.

THEOREM 2.
It is true

$$-\Delta_h E_h(x) = \begin{cases} 1/h^3 & , \quad x = 0 \\ 0 & , \quad x \neq 0 \end{cases} \quad .$$

Notice that the function $E_h(x)$ is an analog to the fundamental solution of the continuous case.

2.2 Some operators

Definition 3. Let $G \subset \mathbb{R}_h^3$ be a bounded connected domain furnished with the property that the boundary ∂G is a subset of the boundary of another domain which is composed by quader stones whose edges lie parallel to the axis.
The translation of the point $x \in \mathbb{R}_h^3$ by $\pm$ h in the x - direction will be denoted by $V^{\pm}_{\alpha,h}x$. Beside, let $f : G \to \mathbb{H}$ be a quaternionic function. It can be represented by

$f(x) = \sum_{i=0}^{3} f_i(x)e_i \ , \ f_i : G \to \mathbb{R}$. The space $L_2(G)$ is given by $\left\{ f : G \to \mathbb{H} \ ; \ \sum_{y\in G} |f(y)|^2 h^3 < \infty \right\}$. A discrete generalization of the Cauchy-Riemann operator of the classical function theory, of the ∇-operator of the hypercomplex function theory, respectively, is defined as follows

$$(D_h^{\pm}(\nabla)f)(x) = \pm \sum_{i=1}^{3} e_i \frac{f(V_{i,h}^{\pm}x)-f(x)}{h} \tag{2.11}$$

A fundamental solution of this operator is determined by

$$e_h^{+}(x) = D_h^{-}(\nabla)\, E_h(x) \quad .$$

Further we define

$$(T_h^{+}f)(x) = \sum_{y\in \overset{\circ}{G}\cup\partial G_l} e_h^{+}(x-y)f(y)h^3 \tag{2.12}$$

and

$$(F_h^{+}f)(x) = T_h^{+}\, D_h^{+}(\nabla)f + f(x) \tag{2.13}$$

where ∂G_l signifies the "left" boundary, this means that $\partial G_l = \{ x\in\partial G : \exists\, \alpha\in\{1,2,3\}$ with $V_{\alpha,h}^{-}x \notin G$. Notice that similarly is introduced the "right" boundary

$$\partial G_r = \{ x\in\partial G : \exists\, \alpha\in\{1,2,3\} \text{ with } V_{\alpha,h}^{+}x \notin G \} .$$

Remark 5.
The operator T_h^{+} is a discrete analog of the hypercomplex T-operator. The relation (2.13) can be seen as a discrete analog of the Borel-Pompeiu formula.

In order to obtain the complete analogy to the continuous case it is necessary to show that the operator F_h^{+} is also defined on the the boundary values of these functions which are given in G. For this purpuse we have to carry out comprehensive calculations, which will be omitted here. Indeed, let us only formulate the result. It consists in the following representation of the operator F_h^{+}

$$(F_h^{+}f)(x) = -\sum_{\alpha} \sum_{\partial G_{r\alpha}\cup\partial G_{l\alpha}} e(x-V_{h,\alpha}^{-}y)D(n)f(y)h^2 \; , \; x\in \overset{\circ}{G}\cup\partial G_l \tag{2.14}$$

where $V^{\pm}_{\alpha,h}x = V^{\pm}_{h,\alpha}x$,

$$\partial G_{1\alpha} = \left\{ x \in \partial G_1 : V^{-}_{\alpha,h}\, x \notin G \right\} ,$$

$$\partial G_{r\alpha} = \left\{ x \in \partial G_r : V^{+}_{\alpha,h}\, x \notin G \right\} ,$$

$D(n) = \sum_{i=1}^{3} n_i e_i$. The vector $n = (n_1, n_2, n_3)$ denoted the unit vector of the outer normal on the surface ∂G.

Remark 6.
Of course it is possible to define F_h^+ by (2.14) whence we can get the Borel-Pompeiu formula (2.13).

Next we intend to investigate the algebraic properties of the operators I, F_h^+ , T_h^+ and $D_h^+(\nabla)$.

Property 1. $(D_h^+(\nabla)T_h^+ f)(x) = f(x) \quad x \in \mathring{G} \cup \partial G_1$ (2.15)

Proof. It holds

$$(D_h^+(\nabla)T_h^+ f)(x) = \sum_{y \in \mathring{G} \cup \partial G_1} D_h^+(\nabla) e_h^+(x-y) f(y) h^3 = f(x) .$$

Property 2. $F_h^+ f \in \ker D_h^+(\nabla)(\mathring{G} \cup \partial G_1)$ (2.16)

Proof. Act $D_h^+(\nabla)$ on $F_h^+ f$. Make use of the formulas (2.13) and (2.15).

Property 3. $F_h^{+2} = F_h^+$ (2.17)

Proof. Note that

$$\left[I - T_h^+ D_h^+(\nabla)\right]\left[I - T_h^+ D_h^+(\nabla)\right] f = \left[I - 2T_h^+ D_h^+(\nabla) + T_h^+ D_h^+(\nabla) T_h^+ D_h^+(\nabla)\right] f =$$

$$= \left[I - T_h^+ D_h^+(\nabla)\right] f \quad , \; x \in \mathring{G} \cup \partial G_1 .$$

Property 4. $(F_h^+ f)(x) = f(x)\,, \quad x \in \mathring{G} \cup \partial G_1, \; \forall f \in \ker D_h^+(\nabla)(\mathring{G} \cup \partial G_1)$ (2.18)

Proof. Make use of (2.13).

Property 5. $T_h^+ f \in \ker D_h^+(\nabla)\left(\text{co}\ (\overset{\circ}{G} \cup \partial G_1)\right)$ (2.19)

Proof. Act $D_h^+(\nabla)$ on T_h^+ .

Property 6. $(T_h^+ D_h^+(\nabla) f)(x) = -(F_h^+ f)(x),\ \forall x \in \text{co}\ (\overset{\circ}{G} \cup \partial G_1)$ (2.20)

Proof. This follows from (2.13).

Property 7. $(F_h^+ f)(x) = 0,\quad x \in \text{co}(\overset{\circ}{G} \cup \partial G_1),\ f \in \ker D_h^+(\nabla)(\overset{\circ}{G} \cup \partial G_1)$ (2.21)

Proof. It is to obtain by virtue of (2.20)

It is an interesting fact, that Property 7. has a conversion. For this reason we assume that f fulfils the relation

$$(F_h^+ f)(x) = 0\ ,\ \forall x \in \text{co}\ (\overset{\circ}{G} \cup \partial G_1)\ .$$

Then

$$(T_h^+ D_h^+(\nabla) f)(x) = 0\ ,\ \forall\ x \in \text{co}\ (\overset{\circ}{G} \cup \partial G_1)$$

and

$$(T_h^+ D_h^+(\nabla) f)(x) = 0\ ,\ \forall\ x \in \partial G\ .$$

Using of (2.13) it is readily seen that

$$(\gamma_0 f)(x) = \gamma_0 (F_h^+ f)(x) \qquad \text{on } \partial G\ .$$

The right side of the latter relation we have to understand as a continuation of $F_h^+ f$ onto ∂G_r , which guarantees that $F_h^+ f \in \ker D_h^+(\nabla)(\overset{\circ}{G})$.

It is well-known that under the above assumptions the boundary value problem

$$-\Delta_h u = 0 \quad \text{in } G$$

$$\gamma_0 u = \gamma_0 f \quad \text{on } \partial G$$

has the unique solution $F_h^+ f$. Hence, Property 7 can be extended to the following equivalence.

Property 7'.

$F_h^+(\gamma_0 f)(x) = 0, \forall x \in co(\overset{\circ}{G} \cup \partial G_1)$ iff existed a function

$g \in \ker D_h^+(\nabla)(\overset{\circ}{G} \cup \partial G_1)$ with $\gamma_0 g = \gamma_0 f$. (2.22)

Property 8. $F_h^+ T_h^+ f = 0$

Proof. Making use of (2.13) it leads to (2.23)

2.3 Applications

The preceding statements were devoted bringing to light several analogies between the continuous and the discrete operator calculus. Now we want to discuss applications of the discrete hypercomplex methods for the approximative solution of boundary value problems of the Laplace equation. Basing on the results of the continuous case it is not difficult to extend these methods to other boundary value problems.

THEOREM 3. (Representation of discrete harmonic functions)
Every function $f \in \ker \Delta_h(\overset{\circ}{G})$ permits the representation

$$f = \Phi_1 + T_h^+ \Phi_2 \qquad (2.23)$$

where $\Phi_1 \in \ker D_h^+(\nabla)(\overset{\circ}{G} \cup \partial G_1)$, $\Phi_2 \in \ker D_h^-(\nabla)(G)$ are uniquely defined functions.

Proof.

Write $f = F_h^+(\gamma_0 f) + T_h^+ D_h^+(\nabla) f$. Then in virtue of (2.13) follows immediately $\Phi_1 = F_h^+(\gamma_0 f)$, $\Phi_2 = D_h^+(\nabla) f$. The uniqueness of the functions Φ_1 and Φ_2 is easy to verify.

In former papers [3],[4] we found that for the treatment of boundary value problems and eigenvalue problems it is very useful to know the complement of the space $L_2(G) \cap \ker D(\nabla)$ and the corresponding orthoprojectors. Therefore we intend to generalize this approach on the discrete case.

Definition 3.

Write $e_h^-(x) = D_h^+(\nabla)E_h(x)$. The operator T_h^- is defined as follows

$$(T_h^- f)(x) = \sum_{y \in \mathring{G} \cup \partial G_r} e_h^-(x-y)f(y)h^3 \quad .$$

THEOREM 4.

The operator

$$\gamma_o T_h^- F_h^+ : \operatorname{im} \gamma_o F_h^+ \longrightarrow \operatorname{im} \gamma_o T_h^- \quad \text{is invertible.}$$

Proof.

By virtue of $T_h^- F_h^+ v \in \ker \Delta_h$ follows from $\gamma_o T_h^- F_h^+ v = 0$ immediately $T_h^- F_h^+ v = 0$ and therefore $F_h^+ v = 0$. The relation (2.13) yields $v = T_h^+ D_h^+(\nabla) v$ as $v = F_h^+ w$ and concluding $v = 0$. It remains to prove that $\gamma_o T_h^- F_h^+$ is a surjection. Let $\gamma_o w \in \operatorname{im} \gamma_o T_h^-$, what means that $\gamma_o w = \gamma_o T_h^- v$.

The boundary value problem

$$\Delta_h w = 0 \quad \text{in } \mathring{G}$$
$$\gamma_o w = \gamma_o T_h^- v \quad \text{on } \partial G$$

has a unique solution and can be represented in a unique way by $w = \phi_1 + T_h^- \phi_2$ where $\phi_1 \in \ker D_h^-(\nabla), \phi_2 \in \ker D_h^+(\nabla)$, whence $w = \phi_1 + T_h^- F_h^+ \phi_2$. From $\gamma_o w = \gamma_o T_h^- v$ one conclude $\phi_1 = 0$ which means that $\gamma_o w = (\gamma_o T_h^- F_h^+)\phi_2$.

THEOREM 5.

The discrete space $L_2(G)$ admits the orthogonal decomposition

$$L_2(G) = \left[\ker D_h^+(\nabla)(\mathring{G} \cup \partial G_1) \cap L_2\right] \oplus D_h^-(\nabla) W_2^1(\mathring{G} \cup \partial G_r).$$

Proof.

First we show that the defined subspaces are orthogonal. Let

$f \in \ker D_h^+(\nabla)(\mathring{G} \cup \partial G_1)$, $g \in D_h^-(\nabla)\mathring{W}_2^1(\mathring{G} \cup \partial G_r)$. This implies $g = D_h^-(\nabla)H$. We have that

$$\langle f,g\rangle_{L_2} = \sum_{y\, G} \overline{f(y)}g(y)h^3 = \sum_{G}\sum_{\alpha} \overline{f(y)}e \;\frac{H(y)-H(V_{\alpha,h}^- y)}{h}\, h^3 =$$

$$= \sum_{\mathring{G}}\sum_{\alpha} \overline{f(y)}e_\alpha H(y)h^2 - \sum_{\alpha}\sum_{V_{\alpha,h}^-\mathring{G}} f(V_{h,\alpha}^+ y) = \sum_{\mathring{G}}\sum_{\alpha} \overline{f(y)}e_\alpha H(y)h^2 -$$

$$- \sum_{\alpha}\sum_{\beta}\sum_{V_{\alpha,h}^- \partial G_{1,\beta} \cup V_{\alpha,h}^- \partial G_{r,\beta}} \overline{f(V_{h,}^+ y)e_\alpha} H(y)h^2 -$$

$$- \sum_{\alpha}\sum_{V_{\alpha,h}^-\mathring{G}} \overline{f(V_{h,\alpha}^+ y)e_\alpha} H(y)h^2 = \sum_{\mathring{G}}\sum_{\alpha} \overline{f(y)e_\alpha} H(y)h^2 -$$

$$- \sum_{\alpha}\sum_{V_{\alpha,h(\mathring{G} \cup \partial G_{r,\alpha})}} \overline{f(V_{h,\alpha}^+ y)e_\alpha} H(y)h^2 = \sum_{\mathring{G}}\sum_{\alpha} \overline{f(y)e_\alpha} H(y)h^2 -$$

$$- \sum\sum_{\mathring{G} \cup \partial G_{1,\alpha}} \overline{f(V_{h,\alpha}^+ y)e_\alpha} H(y)h^2 = \sum_{\mathring{G}} \overline{D_h^+(\nabla)f(y)}H(y)h^3 = 0$$

The explicit construction of the orthoprojectors on the subspaces yields that these are complementary. This is the content of the next theorem.

<u>THEOREM 6.</u>
The discrete operators

$$\mathbb{P}_h^+ = F_h^+ (\gamma_o T_h^- F_h^+)^{-1} \gamma_o T_h^- : \; L_2(G) \longrightarrow L_2(G)$$

$$\mathbb{Q}_h^+ = I - \mathbb{P}_h^+$$

are the orthoprojectors on $\ker D_h^-(\nabla)(\mathring{G} \cup \partial G_1)$ and on $D_h^-(\nabla)\mathring{W}_2^1(\mathring{G} \cup \partial G_r)$.

Proof.

Obviously holds $\operatorname{im}\mathbf{P}_h^+ \subset \ker D_h^+(\nabla)(\mathring{G}\cup\partial G_1)$. As for

$f \in \ker D_h^+(\nabla)(\mathring{G}\cup\partial G_1)$

$$\mathbf{P}_h^+ f = F_h^+(\gamma_o T_h^- F_h^+)^{-1}\gamma_o T_h^- f = F_h^+(\gamma_o T_h^- F_h^+)^{-1}(\gamma_o T_h^- F_h^+) f = F_h^+ f = f$$

we have $\operatorname{im}\mathbf{P}_h^+ = \ker D_h^+(\nabla)(\mathring{G}\cup\partial G_1)$. A simple calculation shows that

$$\mathbf{P}_h^{+2} = \mathbf{P}_h^+ \quad , \quad \mathbf{Q}_h^{+2} = \mathbf{Q}_h^+ \quad , \quad \mathbf{P}_h^+\mathbf{Q}_h^+ = \mathbf{Q}_h^+\mathbf{P}_h^+ = 0 \text{ .}$$

On the other hand $\operatorname{im}\mathbf{Q}_h^+ \subset D_h^-(\nabla)\mathring{W}_2^1(\mathring{G}\cup\partial G_r)$. It is easy to see because

$$T_h^-\mathbf{Q}_h^+ f = T_h^- f - T_h^- F_h^+ (\gamma_o T_h^- F_h^+)^{-1}\gamma_o T_h^- f \in \mathring{W}_2^1$$

whence

$\mathbf{Q}_h^+ f \in D_h^-(\nabla)\mathring{W}_2^1(\mathring{G}\cup\partial G_r)$. Let now $f = D_h^-(\nabla)H$, $H \in \mathring{W}_2^1$. Then it follows

$$\mathbf{Q}_h^+ f = \mathbf{Q}_h^+ D_h^-(\nabla)H = D_h^-(\nabla)H - F_h^+(\gamma_o T_h F_h^+)^{-1}\gamma_o T_h D_h^-(\nabla)H =$$

$$= f - F_h^+(\gamma_o T_h^- F_h^+)^{-1}\gamma_o H = f \text{ .}$$

On this way we obtain explicit representations of the solutions of the problems

$$\begin{aligned} -\Delta_h u &= 0 \\ \gamma_o u &= g \end{aligned} \qquad (2.24) \qquad \text{respectively} \qquad \begin{aligned} -\Delta_h v &= f \\ \gamma_o v &= 0 \end{aligned} \qquad (2.25)$$

.

THEOREM 7.

The solution of the problem (2.24) has the representation

$$u = F_h^- g + T_h^- F_h^+(\gamma_o T_h^- F_h^+)^{-1}\gamma_o(f - F_h^- f) \tag{2.26}$$

and the solution of the problem (2.25) is given in the form

$$v = T_h^-\mathbf{Q}_h^+ T_h^+ f \text{ .} \tag{2.27}$$

Proof. Make use of the decomposition

$$-\Delta_h = D_h^+(\nabla)D_h^-(\nabla) \ .$$

Remark 7.
If we start the above considerations with $D_h^-(\nabla)$ then we obtain similar results.

Remark 8.
Since it is possible to calculate the functions $E_h(x)$, $e_h(x)$ the representations (2.26) and (2.27) are given a numerical meaning. The solution of the discrete boundary value problem is reduced to a matrix multiplication.

Remark 9.
Since the functions $E_h(x)$ and $e_h^+(x)$ are homogeneous with respect to h it is sufficient to compute these functions with a fixed meshwidth. More exactly holds

$$E_{h_2}(ih_2,jh_2,kh_2) = \frac{h_1}{h_2} E_{h_1}(ih_1,jh_1,kh_1)$$

$$e^+_{h_2}(ih_2,jh_2,kh_2) = \left(\frac{h_1}{h_2}\right)^2 \quad e^+_{h_1}(ih_1,jh_1,kh_1)$$

Remark 10.
In order to find out a connection between the solutions of the discrete and continuous boundary value problems we have to consider the behaviour of the described formulas for $h \to 0$. These investigations will be carried out in another paper.

Remark 11.
The formulas for the discrete operators F_h^+ and T_h^+ can be used as suitable formulas of quadrature for the continuous operators F and T .

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KLAUS GÜRLEBECK
TU KARL-MARX-STADT
SEKTION MATHEMATIK
DDR 9010 KARL-MARX-STADT
PSF 964

WOLFGANG SPRÖSSIG
BERGAKADEMIE FREIBERG
SEKTION MATHEMATIK
DDR 9200 FREIBERG
BERNHARDT-VON-COTTA-STR.2