Jaroslav Hrubý

Supersymmetric quantum mechanics and $U(N)$-nonlinear Schrödinger equation

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 16. pp. [65]--70.

Persistent URL: `http://dml.cz/dmlcz/701409`

SUPERSYMMETRIC QUANTUM MECHANICS AND U(N)-NONLINEAR SCHRÖDINGER EQUATION

J. Hrubý

In recent time the application of the supersymmetric quantum mechanics (SSQM) to the vector version of the nonlinear Schrödinger equation, i.e. U(N)-NLS, was presented (HRUBÝ J. and MAKHANKOV V.G.).

The U(N)-NLS has the form

$$i\varphi_{N,t} + \varphi_{Nxx} + (\bar{\varphi}_N \varphi_N)\varphi_N = 0\,, \tag{1}$$

where
$$\varphi_N(x,t) \underset{m}{=} (\varphi_{N,1}, \dots, \varphi_{N,m})^T\,,$$
$$\bar{\varphi}_N \varphi_N = \sum_{j=1} |\varphi_{N,j}|^2\,, \quad m \geq N\,.$$

The eq.(1) has a wide application in physics and the intensive study of this equation was started after the integrability of the U(1) version (ZAKHAROV V.E. and SHABAT A.B.) and then of the vector versions U(N), U(P,Q) was shown.

Here, we show a new possibility to investigate a new class of the soliton solutions of the U(N)-NLS.

A new particular class of the soliton solutions of eq.(1) has been obtained via the so-called factorization method and a technique, in a sense, similar to that developed by Krichever (KRICHEVER I.M.).

We show that these solutions are equivalent to the reflectionless symmetric potentials of the one-dimensional Schrödinger eq.:

$$i\psi_t + \psi_{xx} - U\psi = 0\,, \tag{2}$$

where $\psi(x,t,k)$ is scalar complex function and k is the complex parameter.

In the case when the potentials $U \equiv u_N(x)$ have the form

$$u_N(x) = -N(N+1)\,b^2 \operatorname{sech}^2 bx\,, \tag{3}$$

for N=1,2,...,we show that they correspond to the potentials obtained via SSQM.

We can show this in the following way:

$$\varphi_N(x,t) = C e^{iW}\phi_N(y), \tag{4}$$

where $\phi_N(y) = (\phi_{N,1}, \dots \phi_{N,m})^T$, $C = diag(C_1, \dots, C_m)$

$W = diag(\theta_1, \dots, \theta_m)$, $\theta_j = \frac{v}{2}(x - \frac{v}{2}t) - \Lambda_j t$, $y = x - vt$.

We insert (4) into (1) to get

$$\phi''_{N,j} - u_N \phi_{N,j} = -\Lambda_j \phi_{N,j}, \tag{5}$$

$$u_N = -\sum_{j=1}^{m} |C_j|^2 \phi^2_{N,j}$$

Suppose the potentials u_N to be in the form (3).Then (5) becomes

$$\phi''_{N,j} + N(N+1)b^2 sech^2 by\, \phi_{N,j} = -\Lambda_j \phi_{N,j} . \tag{6}$$

It is well known (KRICHEVER I.M.) that eq. (6) has,for arbitrary N, N eigenvalues $\Lambda_j = -j^2 b^2$, $j = 1, 2, \dots, N$.

The corresponding eigenfunctions may be found by using the factorization which is equivalent to the SSQM "square root",as it is usual:

we can define $A_\ell^\pm$ in the following way

$$A_\ell^\pm = \pm\frac{d}{dy} + v_\ell(y) = \pm\frac{d}{dy} + \ell b\, th\, by$$

If we denote "supercharges" as

$$Q_\ell^- = \begin{pmatrix} 0 & 0 \\ A_\ell^- & 0 \end{pmatrix}, \qquad Q_\ell^+ = \begin{pmatrix} 0 & A_\ell^+ \\ 0 & 0 \end{pmatrix},$$

then the SSQM "superalgebra" has the form:

$$(Q_\ell^-)^2 = (Q_\ell^+)^2 = 0,$$

$$[H_s, Q_\ell^-] = [H_s, Q_\ell^+] = 0,$$

$$\{Q_\ell^-, Q_\ell^+\} = H_s,$$

where $H_s = \begin{pmatrix} A_\ell^+ A_\ell^- & 0 \\ 0 & A_\ell^- A_\ell^+ \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} + v_\ell^2(x) + v_{\ell x}(x) & 0 \\ 0 & -\frac{d^2}{dx^2} + v_\ell^2(x) - v_\ell(x) \end{pmatrix}$

As is usual in SSQM, we define

$$A^{+}_{\ell+1}\,\phi_{\ell,j} = \phi_{\ell+1,j} \tag{7}$$

$$A^{-}_{\ell+1}\,\phi_{\ell,j} = \phi_{\ell-1,j} \tag{8}$$

From (7) and (8) using

$$\phi_{\ell,j} \equiv 0 \tag{9}$$

for $\ell > N$, we obtain all the solutions to eq.(6).

Some of them follow directly:

for $N = j = \ell$ we get

$$A^{+}_{N}\,\phi_{N,N} = 0$$

and from this

$$\phi_{N,N} \sim \operatorname{sech}^{N} by\,. \tag{10}$$

Generally, we have the recurrent formula

$$\phi_{\ell,j} = A^{+}_{\ell} A^{+}_{\ell-1} \cdots A^{+}_{j+1}\,\phi_{j,j}\,. \tag{11}$$

Thus we obtain for $N = 1 = m$, i.e. $\ell = j = 1$ from (10)

$$\phi_{1,1} \sim \operatorname{sech} by\,. \tag{12}$$

For N=2 we have two solutions corresponding to $\Lambda_1 = -b^2,\ \Lambda_2 = -4b^2$. Then, from (10), it follows

$$\phi_{2,2} \sim \operatorname{sech}^2 by \tag{13a}$$

and from (11) and (12) we get

$$\phi_{2,1} = A^{+}_{2}\,\phi_{1,1} \sim \operatorname{th} by \operatorname{sech} by\,. \tag{13b}$$

So, we obtain from the relations (4) and (12) the known one-soliton solution of the U(1)-NLS

$$\varphi_1(x,t) = C\, e^{i\theta_1} \operatorname{sech} by\,, \tag{14}$$

where $|C|^2 = 2b^2$.

For N=m=2, the soliton solution to the U(2)-NLS can be expressed as

$$\varphi_2(x,t) = \begin{pmatrix} C_1\, e^{i\theta_1} \operatorname{sh} by \\ C_2\, e^{i\theta_2} \end{pmatrix} \operatorname{sech}^2 by \tag{15}$$

where $|C_1|^2 = |C_2|^2 = 6b^2$.

Analogously for N=m=3 and so on.

The general expression for the symmetric reflectionless potentials $\mu_N(x)$ in (3) can be given in SSQM following Sukumar (SUKUMAR C.V.):

$$\mu_N = -2\frac{d^2}{dx^2}\ln\det D_N \quad , \tag{16}$$

where the elements of the matrix D_N are given by

$$[D_N]_{jk} = \frac{1}{2}(\mu_k)^{j-1}\left[e^{\mu_k x} + (-1)^{j+k}e^{-\mu_k x}\right] \tag{17}$$

and the normalised eigenfunctions for the eigenenergy may be written in the form

$$\tilde{\varphi}_N(E_j) = \left[\frac{\mu_j}{2}\sum_{k\neq j}^{N}|\mu_k^2-\mu_j^2|\right]^{\frac{1}{2}}\left[D_N^{-1}\right]_{jN} \quad , \tag{18}$$

where j=1,2,...,N.

For N=2, from the relations (16-18) it follows

$$D_2 = \begin{pmatrix} \operatorname{ch}\mu_1 x & \operatorname{sh}\mu_2 x \\ \mu_1 \operatorname{sh}\mu_1 x & \mu_2 \operatorname{ch}\mu_2 x \end{pmatrix} \quad , \tag{19}$$

$$u_2(x) = -2(\mu_2^2-\mu_1^2)\frac{\mu_2^2\operatorname{ch}^2\mu_1 x + \mu_1^2\operatorname{sh}^2\mu_2 x}{(\mu_2\operatorname{ch}\mu_2 x\operatorname{ch}\mu_1 x - \mu_1\operatorname{sh}\mu_2 x\operatorname{sh}\mu_1 x)^2} \quad , \tag{20}$$

$$\tilde{\varphi}_2(E_1) = \left[\frac{\mu_1}{2}(\mu_2^2-\mu_1^2)\right]^{\frac{1}{2}}\frac{\operatorname{sh}\mu_2 x}{\det D_2} \quad , \tag{21}$$

$$\tilde{\varphi}_2(E_2) = \left[\frac{\mu_2}{2}(\mu_2^2-\mu_1^2)\right]^{\frac{1}{2}}\frac{\operatorname{ch}\mu_1 x}{\det D_2} \quad . \tag{22}$$

We now show how this results of SSQM coincide with results in ref.(MAKHANKOV V.G. and MYRZAKULOV R.).

The possibility to use the eq.(2) for constructing the solutions of the eq.(1) is valid from the following:

in the k-plane there exist N points k_j, j=1,2,...,N and the relation

$$u_N(x,t) = -\sum_{j=1}^{N}|C_j|^2|\psi(x,t,k_j)|^2 \quad , \tag{23}$$

where $C_j = \text{const.}$, is valid.

It means, that the functions

$$\varphi_{N,j} = C_j \psi(x,t,k_j)$$

are the solutions U(N) of the eq.(1).

The solution of eq.(2) has the form

$$\psi(x,t,k) = e^{iky - ik^2t + i\alpha}\left(1 + \frac{i\xi_1}{2k} + \frac{\xi_2}{4k^2} + \frac{\xi_3}{k^3} + \dots + \frac{\xi_n}{k^n}\right), \tag{24}$$

where $\alpha = \frac{v}{2}\left(x - \frac{v}{2}t\right)$, $\xi_j = \xi_j(y)$, $y = x - vt$.

We discusss these solutions for m=1,2.When N=1 and $m > 1$ we get the known vector generalization of the solution (14). For N=2 we put (24) in (2) and we get

$$U + \xi_1' = \xi_1'' + \xi_2' - U\xi_1 = \xi_2'' - U\xi_2 = 0 . \tag{25}$$

From (24) follows:

$$\begin{aligned} &\xi_2 + \xi_1' + \frac{1}{2}\xi_1^2 - 2a = \\ &= \frac{1}{2}\xi_2^2 + \xi_1'\xi_2 - \xi_1\xi_2' - 2b = 0, \end{aligned} \tag{26}$$

where a,b=const.

Eliminating ξ_2 from (26) and putting

$$\xi_1 = \frac{2z'}{z}$$

we obtain:

$$z^{(IV)} - 2az'' + (a^2 - b)z = 0 . \tag{27}$$

One solution of eq.(27) has the form

$$z(y) = 2\kappa \operatorname{ch} \nu y + 2\nu \operatorname{ch} \kappa y ,$$

when $a = \frac{1}{2}(\nu^2 + \kappa^2)$, $b = \frac{1}{4}(\nu^2 - \kappa^2)$.

So we get $\psi(x,t,k) = e^{iky - ik^2t + i\alpha}\,\phi(y,k)$,

$$\phi(y,k) = 1 + \frac{i\kappa\nu}{k}\left[\frac{\operatorname{sh}\kappa y + \operatorname{sh}\nu y}{\kappa\operatorname{ch}\nu y + \nu\operatorname{ch}\kappa y}\right] + \frac{\kappa^2 - \nu^2}{4k^2}\left[\frac{\kappa\operatorname{ch}\nu y - \nu\operatorname{ch}\kappa y}{\kappa\operatorname{ch}\nu y + \nu\operatorname{ch}\kappa y}\right],$$

$$u(x,t) = -2\kappa\nu\,\frac{2\kappa\nu + (\kappa^2+\nu^2)\operatorname{ch}\nu y\operatorname{ch}\kappa y - 2\kappa\nu\operatorname{sh}\nu y\operatorname{sh}\kappa y}{(\kappa\operatorname{ch}\nu y + \nu\operatorname{ch}\kappa y)^2} . \tag{28}$$

The function $\psi(x,t,k)$ in the points $k_{1,2} = \pm \frac{i}{2}(\nu \mp \varkappa)$ has the form:

$$\psi(x,t,k_1) = e^{ik_1y - ik_1^2t + i\alpha}(\varkappa-\nu)\frac{\operatorname{ch}\nu y - \operatorname{ch}\varkappa y - \operatorname{sh}\nu y - \operatorname{sh}\varkappa y}{\varkappa\operatorname{ch}\nu y + \nu\operatorname{ch}\varkappa y}, \quad (29)$$

$$\psi(x,t,k_2) = e^{ik_2y - ik_2^2t + i\alpha}(\varkappa+\nu)\frac{\operatorname{sh}\nu y + \operatorname{sh}\varkappa y - \operatorname{ch}\nu y - \operatorname{ch}\varkappa y}{\varkappa\operatorname{ch}\nu y + \nu\operatorname{ch}\varkappa y}. \quad (30)$$

The following is valid:

$$u(x,t) = -|C_1|^2|\psi(x,t,k_1)|^2 - |C_2|^2|\psi(x,t,k_2)|^2$$

where $|C_1|^2 = |C_2|^2 = (8\varkappa\nu)^{-\frac{1}{2}}$.

Formulae (28),(29),(30) coincide exactly with formulae (20),(21),(22) obtained via SSQM after reparametrization $\varkappa = -(\mu_1 - \mu_2)$, $\nu = \mu_1 + \mu_2$.

It can be also shown, that for given N, the number of coefficients is the same as the number of the binomial coefficients in the following expansion:

$$1 \equiv (1 + \operatorname{sh}^2 by)^{n-1} / \operatorname{ch}^{2(n-1)} by.$$

In this short communication we showed the application of SSQM to the U(N)-NLS.

The symmetric reflectionless potentials are obtained here as linear combinations of the eigenvalue solutions.

The symmetric reflectionless SSQM potentials from ref.(SUKUMAR C.V.) and those obtained via familiar factorization method naturally coincide up to reparametrization.

REFERENCES

HRUBÝ J.and MAKHANKOV V.G."On the SSQM and nonlinear equations", Dubna Preprint,December (1986),to be publ.

KRICHEVER I.M."Sov.Funct.Anal.",v.20,n.3 (1986).

MAKHANKOV V.G. and MYRZAKULOV R."New class of soliton solution...", Dubna Preprint P5-86-356 (1986).

SUKUMAR C.V."Supersymmetry,potentials with bound states at arbitrary energies and multi-soliton configurations",J.Phys.A Math.Gen.,19 (1986)2297-2314.

ZAKHAROV V.E. and SHABAT A.B. JETP ,v.61 (1971)118.

J. HRUBÝ ,Institute of Plasma Physics,CSAV,182 11 Prague 8,CSSR