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Note on Stieffel-Whitney classes of flag manifolds


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The Stiefel-Whitney characteristic classes seem to contain quite interesting information on real flag manifolds (cf. e.g. [3], [4], [6]). Let $G(k_1, \ldots, k_r)$ denote the real flag manifold $O(k_1+\ldots+k_r)/O(k_1)\times\ldots\times O(k_r)$, where $k_1, \ldots, k_r (r \geq 2)$ are fixed positive integers. For instance, $G(k_1, k_2)$ is the Grassmann manifold of $k_1$-planes (or $k_2$-planes) in real Euclidean $k_1+k_2$-space.

Recall (cf. [5] for details) that over the manifold $G(k_1, \ldots, k_r)$ one has naturally defined $k_i$-dimensional vector bundles $\gamma_i$ ($i=1, \ldots, r$) with their Whitney sum being trivial bundle. For the tangent bundle one has

\[ TG(k_1, \ldots, k_r) = \bigoplus_{1 \leq i < j \leq r} \gamma_i \otimes \gamma_j. \]

Moreover, by [1], the $\mathbb{Z}_2$-cohomology algebra $H(G(k_1, \ldots, k_r); \mathbb{Z}_2)$ can be identified with

\[ Z_2[w_1(\gamma_1), \ldots, w_{k_1}(\gamma_1), \ldots, w_1(\gamma_r), \ldots, w_{k_r}(\gamma_r)]/J, \]

where $J$ is an ideal determined by single relation $\bigoplus_{i=1}^r w(\gamma_i) = 1$. Here $w(\gamma) = 1 + w_1(\gamma) + w_2(\gamma) + \ldots$ means the total Stiefel-Whitney class of a vector bundle $\gamma$. If $M$ is a smooth closed manifold, one puts as usual $w(M) = w(TM)$.

The main purpose of this short note is to illustrate our introductory observation anew by the following

**THEOREM.** If $r \geq 3$, $k_1 \equiv k_2 \equiv \ldots \equiv k_r (\text{mod } 2)$ and $k_1k_2\ldots k_r > 1$, then $w_3(G(k_1, \ldots, k_r)) \in H(G(k_1, \ldots, k_r); \mathbb{Z}_2)$ does not vanish.

As an application, one gets

**COROLLARY.** If $r \geq 3$, then the flag manifold $G(k_1, \ldots, k_r)$ admits an almost complex structure if and only if $k_1 = k_2 = \ldots = k_r = 1$ and $\dim(G(k_1, \ldots, k_r)) = \binom{r}{2}$ is an even number. 

This paper is in final form and no version of it will be submitted for publication elsewhere.
Namely, it is easily verified that the manifold $G(1,\ldots,1)$ is parallelizable.

Therefore, if its dimension is even, this manifold obviously admits an almost complex structure.

Moreover, in order that a real smooth closed manifold $M$ be almost complex, it is necessary that $M$ be even-dimensional, orientable and also that all the integral Stiefel-Whitney classes $W_{2i-1}(M)\in H^{2i-1}(M;\mathbb{Z})$ be zeros (cf. [7, 41.9]), hence the same be true for $w_{2i-1}(M)\in H^{2i-1}(M;\mathbb{Z}_2)$.

Keeping in mind that $k_1\equiv k_2\equiv\ldots\equiv k_r \pmod{2}$ is equivalent to orientability of $G(k_1,\ldots,k_r)$ (cf. [3]), we get Corollary as a consequence of Theorem indeed.

**Proof of Theorem.** Without loss of generality, we suppose $k_1\leq k_2\leq\ldots\leq k_r$. Hence $k_1k_2\ldots k_r>1$ implies clearly $k_r>2$.

Consider first the case $r=3$. If $k_1\equiv k_2\equiv k_3 \pmod{2}$, we compute from (1) (cf. [3] if needed)

$$w_2(G(k_1,k_2,k_3)) = \left[1 + \binom{k_1}{2} + \binom{k_2}{2}\right]w_1^2(\gamma_1) + \left[1 + \binom{k_2}{2} + \binom{k_3}{2}\right]w_1^2(\gamma_2) + w_1(\gamma_1)w_1(\gamma_2).$$

Since $w_1(G(k_1,k_2,k_3))$ is now zero, the Wu formula yields

$$w_3(G(k_1,k_2,k_3)) = w_1^2(\gamma_1)w_1(\gamma_2) + w_1(\gamma_1)w_1^2(\gamma_2).$$

By direct finding a basis in $H(G(k_1,k_2,k_3);\mathbb{Z}_2)$ or by applying the Leray-Hirsch Theorem to the obvious differentiable fibre bundle $G(k_2,k_3)\to G(k_1,k_2,k_3)$

$\xrightarrow{i} \quad G(k_1,k_2,k_3)$

one proves the assertion.

Now recall ([2]) that when $F\subset E$ is a differentiable fibre bundle, then one has $TE = p^*TB\oplus\eta$, where $\eta$ is the "tangent bundle along the fibres". So, if $F$ is connected, $w_1(F)\neq 0$ implies $w_j(E)\neq 0$.

This, when applied to the fibre bundle $G(k_2,\ldots,k_r)\to G(k_1,\ldots,k_r)$

$\xrightarrow{i} \quad G(k_1,k_2,\ldots,k_r), \quad G(k_1,\ldots,k_r) \to G(k_1,\ldots,k_r)$
with an obvious induction, proves Theorem completely.

REFERENCES

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