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SOME GENERALIZATION OF GODEMENT'S THEOREM ON DIVISION

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ABSTRACT. Some generalization of Godement's theorem on division is found. This generalization characterizes all equivalence relation R (on a C^{∞}-manifold) such that every abstract class of R has a countable number of arcwise connected components and the family of all such components is a foliation. Using it, another proof of that classical Godement's theorem is obtained.

The classical Godement's theorem on division [3] - which characterizes regular equivalence relations R on a C^{∞}-manifold V - is well known:

THEOREM 1. (Godement [3]). Let $\dim V=n$. The following conditions are equivalent:

(1) In the set $V/_R$ there exists a differential structure of an n-k-dim. C^{∞}-manifold (with the quotient topology), such that the natural projection $V \to V/_R$ is a submersion.

(2) (a) $R \subset V\times V$ is a proper n+k-dim. C^{∞}-submanifold of $V\times V$,

(b) $pr_1 : R \to V$, $(x,y) \mapsto x$, is a submersion. ∎

The family $\mathcal{L}$ of all abstract classes of an equivalence relation R fulfilling (1) has the following properties:

(1^o) every abstract class of R has a countable number of arcwise connected components,

(2^o) the family $\mathcal{F}$ of all arcwise connected components of all abstract classes of R is a k-dim. foliation.

Of course, here: each arcwise connected component is equal to a connected component.

Now, we give some generalization of Godement's theorem which characterizes all equivalence relations fulfilling (1^o) and (2^o) (in particular, all foliations).

THEOREM 2. Let R be any equivalence relation on a Hausdorff C^{∞}-manifold V with a countable basis. The following conditions are equivalent:

This paper is in final form and no version of it will be submitted for publication elsewhere.

(1) the family $\mathcal{L}$ of all abstract classes of R has the above properties (1^o) and (2^o),

(2) there exists a subset $\Omega \subset R$ such that

(i) $\Delta \subset \Omega$ where $\Delta = \{(x,x);\ x\in V\}$,

(ii) Ω is a proper n+k-dim. C^{∞}-submanifold of $V\times V$,

(iii) $pr_1|\Omega : \Omega \to V$ is a submersion,

(iv) if we denote, for $(x,y)\in R$,
$R_x := R\cap(\{x\}\times V)$, $\Omega_x := (pr_1|\Omega)^{-1}(x)$, $D_{(x,y)}: R_y \to R_x$, $(y,t)\mapsto(x,t)$, then we have that the set $D_{(x,y)}[\Omega_y]\cap\Omega_x$ is open in the man. Ω_x,

(v) the manifolds $\tilde{R}_x$ (see lemma below) have a countable number of connected components.

LEMMA. If $\Omega\subset R$ has properties (i)÷(iv), then, for each point $x\in V$, there exists exactly one C^{∞}-manifold $\tilde{R}_x$ with the set of points R_x, such that, for each $(x,y)\in R_x$,

(a) $D_{(x,y)}[\Omega_y] \subset\subset \tilde{R}_x$ (i.e. is open in $\tilde{R}_x$),

(b) $D_{(x,y)}|\Omega_y : \Omega_y \to D_{(x,y)}[\Omega_y]\subset\subset\tilde{R}_x$ is a diffeomorphism.

The manifolds $\tilde{R}_x$ have the properties:

(i) $D_{(x,y)}: \tilde{R}_y \to \tilde{R}_x$ is a diffeomorphism,

(ii) $\tilde{R}_x \hookrightarrow V\times V$ is an immersion,

(iii) $\tilde{R}_x$ are Hausdorff,

(iv) if, in addition, the family $\mathcal{F}$ of all arcwise connected components of all abstract classes of R is a k-dim. foliation, then the mapping $\gamma_x : L_x \to \tilde{R}_x$, $y\mapsto(x,y)$, is a diffeomorphism for each $x\in V$ (L_x - the abstract class of R through x equipped with the uniquely determined differential structure of an immerse submanifold of V such that each element of $\mathcal{F}$ contained in L_x is an open subman. of L_x).

The very simple proof of this lemma is omitted.

Proof of theorem 2. (1)⇒(2). Let us take any nice covering $\{(U_i,\varphi_i,\mathbb{R}^n);\ i\in N\}$ of $\mathcal{F}$ [2,p.188] and denote by Q^i_x the plaque of the chart (U_i,φ_i) which contains x, $x\in U_i$. Of course, the covering $\{U_i,\ i\in N\}$ of V has the property:

(*) if $x,y\in U_i\cap U_j$ and $y\in Q^i_x$, then $y\in Q^j_x$, $i,j\in N$.

We put
$$\Omega_i := \{(x,y)\in V\times V;\ x\in U_i,\ y\in Q^i_x\} \quad\text{and}\quad \Omega := \bigcup_{i\in N}\Omega_i .$$

We prove that Ω has properties (i)÷(v). (i) is evident. To prove (ii), it suffices to show that

1^o) Ω_i is open in Ω (with respect to the topology induced from $V\times V$),

2^o) Ω_i is a proper C^{∞}-submanifold of $V\times V$.

1^o) results from the equality $\Omega_i = \Omega\cap(U_i\times U_i)$ which is a consequence of (*). To show 2^o), we first define, for each chart (U_i,φ_i), the mappings φ_i^1 and φ_i^2 in such a way that

$$\varphi_i = (\varphi_i^1, \varphi_i^2) = (x \longmapsto (\varphi_i^1(x), \varphi_i^2(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k}).$$

Next, we put

$$v_i : \Omega_i \longrightarrow \mathbb{R}^{n+k},\ (x,y) \longmapsto (\varphi_i^1(x),\ \varphi_i^2(x),\ \varphi_i^1(y)).$$

The inverse mapping of v_i is

$$w_i : \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow \Omega_i,\ (a,b,c) \longmapsto (\varphi_i^{-1}(a,b),\ \varphi_i^{-1}(c,b)).$$

Now, it is easy to see that 2^o) holds (v_i is a global chart on Ω_i). To show condition (iii), it is enough to consider the following commuting diagram

$$\begin{array}{ccc} \Omega_i & \xrightarrow[\approx]{v_i} & \mathbb{R}^n \times \mathbb{R}^k \\ {\scriptstyle pr_1 | \Omega_i}\downarrow & & \downarrow{\scriptstyle pr_1} \\ U_i & \xrightarrow[\approx]{\varphi_i} & \mathbb{R}^n \end{array}$$

for each $i \in N$. To notice condition (iv), we write $\Omega_x = \{x\} \times \bigcup_{i \in N_x} Q_x^i$ where $N_x := \{i \in N;\ x \in U_i\}$. Therefore

$$D_{(x,y)}[\Omega_y] \cap \Omega_x = \bigcup_{j \in N_y,\, i \in N_x} (\{x\} \times (Q_y^j \cap Q_x^i)) \subset \Omega_x.$$

Condition (v) follows from property (iv) of $\tilde{R}_x$ from our lemma.

(2)⇒(1). We assume that $\Omega \subset R$ fulfils (i)÷(v). Let us take the embedding $\hat{u} : V \longrightarrow \Omega$, $x \longmapsto (x,x)$. Of course, $\hat{u}^* T^\alpha \Omega$ (where $T^\alpha \Omega = \operatorname{Ker} \alpha_*$, $\alpha = pr_1 | \Omega$) is a vector bundle of rank k over V. We define a strong homomorphism $\varkappa$ of vector bundles as a superposition

$$\begin{array}{ccccccccc} \hat{u}^* T^\alpha \Omega & \longrightarrow & T^\alpha \Omega & \longrightarrow & T\Omega & \longrightarrow & T(V \times V) & \xrightarrow{(pr_2)_*} & TV \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow{\hat{u}} & \Omega & = & \Omega & \hookrightarrow & V \times V & \xrightarrow{pr_2} & V. \end{array}$$

$\varkappa$ is a monomorphism because, for $x \in V$, $\varkappa_{|x}(v) = (j_x)_*(v)$, $v \in T_{(x,x)} \Omega_x$, where $j_x : \{x\} \times V \xrightarrow{\approx} V$, $(x,y) \longmapsto y$. Thus $E := \operatorname{Im} \varkappa \subset TV$ is a vector subbundle of order k of TV, and $E_{|x} = (j_x)_{*(x,x)} [T_{(x,x)} \tilde{R}_x]$, $x \in V$.

Via bijections $\gamma_x : L_x \to \tilde{R}_x$, $y \longmapsto (x,y)$, $x \in V$, every abstract class of R is equipped with a differential structure of a manifold. The correctness follows from property (i) of the manifolds $\tilde{R}_x$ (see lemma). The manifolds obtained are integral for the distribution E. Indeed, for $x \in V$, the inclusion $L_x \hookrightarrow V$ is an immersion (because it is the superposition $L_x \xrightarrow{\gamma_x} \tilde{R}_x \hookrightarrow \{x\} \times V \xrightarrow{j_x} V$), and $T_x L_x = (j_x)_{*(x,x)} [T_{(x,x)} \tilde{R}_x] = E_{|x}$. Let $\mathcal{F}$ be the family of all connected components of all manifolds L_x obtained above. By the Frobenius' theorem [1,p.86], $\mathcal{F}$ is a k-dim. foliation. To conclude this theorem, we need to demonstrate that the family $\mathcal{F}$ is equal to the family of arcwise connected components of all abstract classes of R. For the purpose, it is sufficient to show that every manifold L_x is a k-leaf of V with respect to all locally arcwise connected topological spaces, i.e. if X is such a space and $f : X \longrightarrow V$ a continuous mapping such that $f[X] \subset L_x$, then the induced mapping $f : X \longrightarrow L_x$ is continuous, too. Let X and f be as above; take $t\in X$ and (U,φ) - a chart around $y:=f(t)$ distinguished by $\mathcal{F}$, $\varphi:U \longrightarrow \mathbb{R}^k\times\mathbb{R}^{n-k}$. Let Q be an arcwise connected component of $U\cap L_x$ through y(with respect to the topology induced from V). Q contains countably many plaques of the chart (U,φ) since L_x has - by (v) - countably many connected components, and each of them - as a connected immerse submanifold of V - has a countable basis. Thus $pr_2[\varphi[Q]]$ is an arcwise connected and countable set in $\mathbb{R}^{n-k}$, so it is one-point. This states that Q is equal to one plaque of the chart (U,φ). The set $f^{-1}[U\cap L_x]$ is open in X. Let B be the arcwise connected component of the set $f^{-1}[U\cap L_x]$, containing x. Of course, B is open in X, $f[B]\subset Q$ and $f|B:B \longrightarrow V_{|Q}=L_{x|Q}$ is continuous. The free choice of $t\in X$ implies the continuity of $f:X \longrightarrow L_x$. ■

REMARK. The connectivity of the manifolds $\tilde{R}_x$ is equivalent (in the above theorem) to the fact that $\mathcal{L}$ is a foliation(i.e. every abstract class of R is an arcwise connected set). ■

REMARK. If Ψ is the subgroupoid (of the groupoid determined by R) generated by the set Ω fulfilling conditions (i)÷(iv) from theorem 2, then the set $\Psi\cap(\{x\}\times V)$, $x\in V$, ia an open-closed subset of $\tilde{R}_x$. ■

THEOREM 3. The following conditions are equivalent:

(1) the family of all abstract classes of R is a k-dim. foliation,

(2) there exists a subset $\Omega\subset R$ such that

(i)÷(iv) as in theorem 2,

(v′) Ω generates R (as a groupoid),

(vi′) the manifolds Ω_x are connected.

Proof. (1)⇒(2). The set Ω constructed in the proof of theorem 2 fulfils (vi′) in an evident manner. The connectedness of manifolds $\tilde{R}_x$ implies that (v′) follows from the last remark.

(2)⇒(1). It suffices to show that the manifolds $\tilde{R}_x$ are connected. Let us take any $x_0\in V$ and $y\in L_{x_0}$. Since Ω generates R, there exist points $x_1,\dots,x_{n-1}\in L_{x_0}$ such that (in the groupoid R) $(x_0,y)= =(x_{n-1},y)\cdot\ldots\cdot(x_1,x_2)\cdot(x_0,x_1)$ where $(x_i,x_{i+1})\in\Omega_{x_i}$, $i=0,\dots,n-1$, $x_n=y$. (vi′) implies the existence of curves $c_i:\langle 0,1\rangle \longrightarrow \Omega_{x_i}$ such that $c_i(0)=(x_i,x_i)$, $c_i(1)=(x_i,x_{i+1})$. We define a curve $c:\langle 0,n\rangle \longrightarrow \tilde{R}_{x_0}$ by the formula $c(t)=D_{(x_0,x_i)}(c_i(t-i))$ for $i\leqslant t\leqslant i+1$ to obtain a continuous curve joining (x_0,x_0) and (x_0,y). ■

Theorem 2 enables us to carry out another proof of the classical Godement's theorem.

Proof of theorem 1. (1)⇒(2) as in Godement's proof.

(2)⇒(1). Let us suppose (a) and (b). We notice that $\Omega:=R$ fulfils properties (i)÷(v) from assertion (2) in theorem 2. Thus, theorem 2 states that the family $\mathcal{F}$ of all connected components of all abstract classes of R is a k-dim. foliation.

Now, we prove that, for each point $x\in V$, there exist a real number $a>0$ and a chart (U,φ) around x distinguished by $\mathcal{F}$, such that

(i) $\varphi:U \xrightarrow{\cong} \mathbb{R}^k\times K(a)$ where $K(a):=\prod^{n-k}(-a,a)$,

(ii) $\varphi(x)=(0,0)$,

(iii) if L is an abstract class of R and $L\cap U\neq\emptyset$, then $L\cap U$ is exactly one plaque of the chart (U,φ).

Let us assume to the contrary that there exists a point $x_0\in V$ such that, for each real number $a>0$ and each chart (U,φ) around x_0 distinguished by $\mathcal{F}$, fulfilling (i) and (ii), we have: there is an abstract class L of R such that the set $L\cap U$ contains at least two different plaques. Take any chart (U,φ) around x_0 distinguished by $\mathcal{F}$ such that $\varphi:U \xrightarrow{\cong} \mathbb{R}^k\times\mathbb{R}^{n-k}$ and $\varphi(x_0)=(0,0)$. Let us set

$$U_m:=\varphi^{-1}[\mathbb{R}^k\times\prod^{n-k}(-\tfrac{1}{m},\tfrac{1}{m})],\quad m\in\mathbb{N}.$$

Of course, $(U_m,\varphi|U_m)$ is a chart distinguished by $\mathcal{F}$, too. Then we find an abstract class L_m such that $L_m\cap U_m$ contains two plaques Q_m^1 and Q_m^2, say $Q_m^1:=\varphi^{-1}[\mathbb{R}^k\times\{c_m^1\}]$, $Q_m^2:=\varphi^{-1}[\mathbb{R}^k\times\{c_m^2\}]$, for some $c_m^1\neq c_m^2$. Let us put $x_m^s:=\varphi^{-1}(0,c_m^s)$, $s=1,2$. Of course, $x_m^s\in L_m$, which means that $(x_m^1,x_m^2)\in R$. Besides $x_m^s \xrightarrow[m\to\infty]{} x_0$, $s=1,2$. Take

$$\Omega':=\{(x,y)\in V\times V;\ x\in U \text{ and } y\in Q_x\}$$

where Q_x denotes the plaque of (U,φ) through x. We prove that Ω' is open in R. First, we note (as about Ω_i in the proof of theorem 2) that Ω' is a proper n+k-dim. C^∞-submanifold of $V\times V$. Thus Ω' is an n+k-dim. proper submanifold of the n+k-dim. manifold R, so it is open in R. Further, since $(x_m^1,x_m^2)\notin\Omega'$ and $(x_m^1,x_m^2) \xrightarrow[m\to\infty]{} (x_0,x_0)$, therefore $(x_0,x_0)\in\Omega'$, which leads to a contradiction because $x_0 \in Q_{x_0}$ implies $(x_0,x_0)\in\Omega$.

From the above it follows that there exists a C^∞-atlas on V consisting of some chart (U,φ) distinguished by $\mathcal{F}$ such that (i)÷(iii) hold for $a=a_\varphi$. Let $\mathcal{A}$ be such an atlas. With the help of $\mathcal{A}$, we shall construct a C^∞-atlas on the topological space $V/_R$, such that the projection $\pi:V\longrightarrow V/_R$ is a submersion. First, from the equality $\pi^{-1}[\pi[U]]=pr_2[pr_1^{-1}[U]]$, $U\subset V$, we get the openess of the projection π. Next, taking a chart $(U,\varphi)\in\mathcal{A}$, we define $\tilde{\varphi}:\tilde{U}\longrightarrow K(a_\varphi)$, where $\tilde{U}=\pi[U]$, in such a way that the diagram

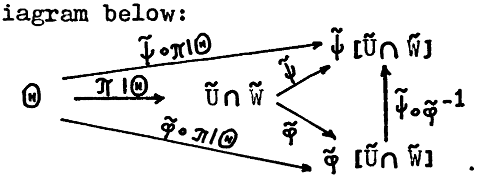
$$(*)\qquad \begin{array}{ccc} U & \xrightarrow{\ \pi\ } & \tilde{U} \\ \downarrow\varphi & & \downarrow\tilde{\varphi} \\ \mathbb{R}^k\times K(a_\varphi) & \xrightarrow{\ p_2\ } & K(a_\varphi), \end{array}\qquad p_2(x,y)=y,$$

comutes. Of course, we must put $\tilde{\varphi}(L):=p_2(\varphi(x))$ for $x\in U\cap L$, $L\in\tilde{U}$. The correctness follows from the fact that $U\cap L$ contains exactly one plaque. The continuity of $\tilde{\varphi}$ follows from the equality $\tilde{\varphi}^{-1}[A]=$

$=\pi[\varphi^{-1}[\mathbb{R}^k\times A]]$, $A\subset K(a_\varphi)$, whereas the openness - from $\tilde{\varphi}[B]=p_2[\varphi[\pi^{-1}[B]\cap U]]$, $B\subset\tilde{U}$. The bijectivity of $\tilde{\varphi}$ is evident. In the end, we take two charts (U,φ) and $(W,\psi)\in\mathcal{A}$ such that $\tilde{U}\cap\tilde{W}\neq\emptyset$. We prove that $\tilde{\psi}\circ\tilde{\varphi}^{-1}$ is of C^∞-class. For the purpose, we put $\Theta:=\pi^{-1}[\tilde{U}\cap\tilde{W}]\subset V$. We notice that Θ is saturated by abstract classes of R, and $\pi[\Theta]=\pi[\Theta\cap U]=\pi[\Theta\cap W]=\tilde{U}\cap\tilde{W}$. Now, we prove - auxiliarily - that $\tilde{\varphi}|\tilde{U}\cap\tilde{W}\circ\pi|\Theta: V_{|\Theta}\to\tilde{\varphi}[\tilde{U}\cap\tilde{W}]\subset\mathbb{R}^{n-k}$ is a submersion. In order to do this, we consider the diagram

$$(**)\qquad \begin{array}{ccc} ((\Theta\cap U)\times V)\cap R & \xrightarrow[pr_1]{sub.} & \Theta\cap U \\ pr_2\big\downarrow sub. & & sub.\big\downarrow \tilde{\varphi}\circ\pi|\Theta\cap U \\ \Theta & \xrightarrow{\tilde{\varphi}\circ\pi|\Theta} & \tilde{\varphi}[\tilde{U}\cap\tilde{W}] \end{array}\qquad \begin{array}{l}(sub.\\ =submersion)\end{array}$$

From $(*)$ we get the submersivity of $\tilde{\varphi}\circ\pi|\Theta\cap U=\tilde{\varphi}|\tilde{U}\cap\tilde{W}\circ\pi|\Theta\cap U$, whereas from diagram $(**)$ - the submersivity of $\tilde{\varphi}\circ\pi|\Theta$. Changing φ to ψ, we get the smoothness of $\tilde{\psi}\circ\pi|\Theta$. To prove that of $\tilde{\psi}\circ\tilde{\varphi}^{-1}$, it is sufficient to analyse the diagram below:



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