Jan Kubarski Pradines-type groupoids

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PRADINES-TYPE GROUPOIDS

Jan Kubarski

<u>ABSTRACT</u>. This paper is devoted to applications of the theory of differential spaces in the sense of R.Sikorski to groupoids. By using these spaces, the notion of a smooth groupoid, much more general than a differential groupoid, is defined here. The theory of foliations is the source of such groupoids. Next, J.Pradines' idea of constructing, for every diff. groupoid, some vector bundle with natural algebraic structures - called the Lie algebroid of this diff. groupoid - is used for smooth groupoids.

INTRODUCTION. The notion of a differential groupoid introduced by Ch.Ehresmann [3] is a natural extension of the notion of a Lie group Dif 'erential groupoids (especially Lie groupoids) constitute an appropriate direction for the development of certain geometric theories such as connexions and Lie pseudogroups. The works by J.Pradines [11] : [15] were the landmark in the theory of diff. groupoids. The author defined, for each differential groupoid Φ (over a manifold V) some object - called the Lie algebroid of Φ - which is a vector bundle $T_0^{\alpha} \phi$ over V such that $(T_0^{\alpha} \phi)_{IX} = T_{u_X} \phi_X$, $\phi_X = \alpha^{-1}(x)$, veV, α -the source, u_X - the unit over x. $T_0^{\alpha} \phi$ has the property: there exists some natural bijection between the module of global smooth sections of this bundle and the module of smooth right-invariant vector fields on $\pmb{\Phi}$. It enables one to carry an R-Lie algebra structure to the module of all global sections of the bundle $T^{\alpha}_{O}\phi$. This notion generalizes the notion of a Lie algebra of a Lie group. Some new directions of the development of the theory of groupoids are described by J.Pradines in [16]. The theory of foliations (also that of pseudogroups) is the source of the important nontransitive groupoids whose space_is not - in general - a manifold. For example, the subgroupoid Φ^{5} of a Lie groupoid $\mathbf{\Phi}$ (over V) consisting of the elements for which the source and the target lie on some leaf of a given foliation for V. However, it is evident that one can always define on $\Phi^{\mathcal{F}}$ some natural structu-

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re of a differential space in the sense of R.Sikorski [19] (see also [20],[21]), and it turns out that all operations in \oint^{T} are then smooth as the mappings in the category of differential spaces. This gives rise to the defining of the notion of a groupoid in the category of diff. spaces. If, in addition, the sets $\alpha^{-1}(x)$, $x \in V$ (α - the source in a given groupoid Φ over V) are the so-called leaves of the diff. space Φ , then this groupoid is called a smooth groupoid. Φ^{T} is such a groupoid.

The present author's observation show that, by following the idea of J.Pradines, one can construct, for each smooth groupoid Φ , an object A(Φ), analogous to the Lie algebroid of a diff. groupoid, not being - unfortunately in general - a vector bundle. The above example Φ^{Φ} of a smooth groupoid have the property that the constructed object A(Φ^{*}) is a vector bundle (although Φ^{*} is hardly ever diff. groupoid). The smooth groupoids Φ for which the objects A(Φ) are vector bundles shall be call the <u>Pradines-type groupoids</u>. An especially important role will be played by those groupoids from among them which are also the so-called smooth groupoids over foliations. They are - in the author's opinion - a proper generalization of principal fibre bundles, for they enable one to build a sensible theory of connexions (see [81, [91).

This work (in the considerable part) has come into being on the basis of preprint [7] and is its extension.

<u>1. PRELIMINARIES</u>. First, we give two definitions fundamental for our work: of a groupoid and of a differential space.

By a <u>groupoid</u> we shall mean (after N.V.Que [171) the system (1.1) $(\Phi, \alpha, \beta, V, \cdot)$ consisting of sets Φ and V and mappings $\alpha, \beta: \Phi \rightarrow V$, $\cdot: \Phi * \Phi \rightarrow \Phi$ where re $\Phi * \Phi = \{(g,h) \in \Phi \times \Phi; \alpha g = \beta h\}$, fulfilling the axioms (i) $\alpha(g \cdot h) = \alpha h$ and $\beta(g \cdot h) = \beta g$ for $(g,h) \in \Phi * \Phi$, (ii) $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for (f,g), (g,h) $e \Phi * \Phi$, (iii) for each point xeV, there exists an element $u_x \in \Phi$ such that $\alpha(u_x) = \beta(u_x) = x$, $h \cdot u_x = h$ when $\alpha h = x$, $u_x \cdot g = g$ when $\beta g = x$ (u_x is uniquely determined and called the <u>unit over x</u>), (iv) for each element h $e \Phi$, there exists an element $h^{-1}e \Phi$ such that $\alpha(h^{-1}) = \beta h$, $\beta(h^{-1}) = \alpha h$, $h \cdot h^{-1} = u_{\beta h}$, $h^{-1} \cdot h = u_{\alpha h}$ (h^{-1} is uniquely determined).

By a <u>differential space</u> (d.s. for short) (see R.Sikorski [19]:[21] we mean each couple (M,C) (sometimes denoted briefly by M) consisting of a set M and a non-empty family C of real functions on M closed with respect to (w.r.t.) <u>localization</u> and <u>superposition with all functions of C^{OD}-class on the Catesian spaces</u>. The set C is then called

the differential structure of this space (d.str. for short) and M its <u>support</u>. More precisely, let us denote by $\tau_{\rm C}$ the weakest topology on M such that all functions of C are continuous. For ACM, we denote by $C_{\rm A}$ the set of all functions h:A $\rightarrow \mathbb{R}$ such that, for any x ϵA , there exist a neighbourhood (nbh) U $\epsilon \tau_{\rm C}$ of x and a function geC satisfying hIUOA=gIUOA. The closedness w.r.t. localization may be expressed in the form $C_{\rm M}$ =C. Denote by scC the set of all functions $\varphi(g_1(\cdot),\ldots,g_{\rm m}(\cdot))$ where φ is a C^O-function on $\mathbb{R}^{\rm m}$, $g_1 \epsilon C$, $i \leq m$, m=1,2,. ..The closedness w.r.t. superposition with all functions of C^O-class on the Cartesian spaces means that scC=C.

Every d.s. (M,C) is also considered as the topological space (M, τ_{C}). If \hat{C} is a non-empty family of real functions on M, then C:= =(sc \hat{C})_M is the smallest d.str. on M containing \hat{C} . C is called the <u>d.str. generated by \hat{C} .</u> If (M,C) is a d.s., then (A,C_A) is such a space, too, for any subset ACM and is called a <u>proper differential sub-</u> <u>space of (M,C)</u> (proper d.subs. for short). C_A is called <u>induced from</u> (M,C) on A. (A,C_A) is sometimes denoted by M_{1A}.

Let (M,C) and (N,D) be any d.s.'s. The mapping $f:M \to N$ is called (i) <u>smooth</u> if gofeC for geD. Then we write $f:(M,C) \to (N,D)$, (ii) a <u>diffeomorphism</u> (diff. for short) if it is a bijection and f and f^{-1} are smooth, (iii) an <u>embedding</u> if $f:(M,C) \to (f[M],D_{f[M]})$ is a diff. By a <u>product</u> $(M,C)\times(N,D)$ we mean the d.s. $(M\times N,C*D)$ where C*D is the d.str. generated by $\{g \circ pr_1, g \in C\} \cup \{h \circ pr_2, h \in D\}$. For ACM and BCN we have $(C*D)_{A\times B} = C_A \times D_B$.

have $(\mathbb{C}\times\mathbb{D})_{A\times B} = \mathbb{C}_A \times \mathbb{D}_B$. Let V be any C^O-manifold. Then $(V, \mathbb{C}^{O}(V))$ is, of course, a d.s. and $\tau_{\mathbb{C}}^{O}(V) \subset \text{Top } V$. What is more, the equality $\tau_{\mathbb{C}}^{O}(V) = \text{Top } V$ holds if and only if (iff) V is Hausdorff. In the Hausdorff case, for any open set U C V (the notation: UC V), we have $\mathbb{C}^{O}(V_{|U}) = \mathbb{C}^{O}(V)_{U}$. In this connection, we adopt the following definition [19]: a d.s. (V,C) is called an <u>n-dim. differential manifold</u> (d.man.) if, for each point xeV, there exist a nbh Uet_C of x, an open subset $\Omega \subset \mathbb{R}^n$ and some diff. $q:(U,C_U) \to (\Omega, C^{O}(\mathbb{R}^n)_{\Omega})$. The topology τ_C is then Hausdorff. We shall identify a Hausdorff \mathbb{C}^{O} -manifold V with the d.man. $(V, \mathbb{C}^{O}(V))$.

Having d.s.'s at our disposal, we are able to give the following (1.2) DEFINITION. By a groupoid in the category of d.s.'s we mean groupoid (1.1) in which Φ and V are d.s.'s and the mappings $\ll, \beta: \Phi \rightarrow V^{-1}: \Phi \rightarrow \Phi$, $h \mapsto h^{-1}$, $u: V \rightarrow \Phi$, $x \mapsto u_x$, as well as $\cdot: \Phi \ast \Phi \rightarrow \Phi$ are smooth ($\Phi \ast \Phi$ denotes here the proper d.subs. of $\Phi \times \Phi$).

We notice that $u: \mathbb{V} \longrightarrow \Phi$ is an embedding.

From now, by a <u>groupoid</u> we shall mean a groupoid in the category of d.s.'s and we shall sometimes say "a groupoid Φ " instead of "a

groupoid $(\Phi, \alpha, \beta, V, \cdot)$ ".

Homomorphisms between groupoids are defined in an evident manner. (1.3) EXAMPLE. Let $R \subset V \times V$ be any equivalence relation on a d.s. V. Then the system

(1.4) $(R, pr_1 | R, pr_2 | R, V, \cdot),$ where R is here the proper d.subs. of VxV and $(y,z) \cdot (x,y) = (x,z)$, is a groupoid called the groupoid of the equivalence relation R. \blacksquare (1.5) EXAMPLE. Let Γ be any pseudogroup of smooth transformations on a d.man. V. Then, for each k=1,2,..., the set of jets $\{j_x^k f; fe \Gamma, xeD_f\} \subset J^k(V,V)$, with the d.str. induced from $J^k(V,V)$, forms a groupoid.

Groupoid (1.1) is said to be a <u>differential groupoid</u> [3] if Φ and V are d.man.'s and $\alpha, \beta: \Phi \longrightarrow V$ are submersions.Adiff. groupoid is said to be a <u>Lie groupoid</u> [17] if it is transitive. A principal fibre bundle P determines the <u>Lie groupoid of Ehresmann</u> PP⁻¹ [2].

<u>2. SMOCTH GROUPOIDS</u>. The notion of a subspace of a d.s. can be found in [19] but it is too strong for us, so we adopt the following:

A d.s. (N',D') is said to be a <u>differential subspace</u> (d.subs. for short) of a d.s. (N,D) if N'CN, and for each point yeN', there exists a nbh UeT_{D'} of y such that $D'_U=D_U$. Then we write $(N',D') \hookrightarrow (N,D)$. If $D'=D_{N'}$, then (N',D') is a proper d.subs. of (N,D).

Any immerse d.subman. of a d.man. is an example of a d.subs.

By a <u>leaf</u> (<u>k-leaf</u>) of a d.s. (N,C) we mean a subset LCM if there exists a d.str. D on L such that (L,D) is a d.man. (of dimension k), (L,D) is a d.subs. of (M,C) and, for each locally arcwise connected topological space X and a continuous mapping $f:X \rightarrow (M,\tau_C)$ such that $f[X] \subset L$, the induced mapping $\hat{f}:X \rightarrow (L,\tau_D)$, $x \mapsto f(x)$, is continuous, too. We notice that

(1) the d.str. D on L is uniquely determined,

(2) each connected component of (L,D) is equal to an arcwise connected component of the subset L in $(\mathbb{M}, \mathcal{T}_{c})$,

(3) L is a leaf iff any of its arcwise connected components is a leaf,

(4) if (X,E) is any d.s. whose topology τ_E is locally arcwise connected, then, for each smooth mapping $f:(X,E) \rightarrow (M,C)$ such that $f(X) \subset L$, the mapping $\hat{f}:(X,E) \rightarrow (L,D)$, $x \mapsto f(x)$, is also smooth.

Sometimes, the manifold (L,D) will be called a <u>leaf</u> of (M,C). Each element of a foliation of a d.man. is an example of a leaf. Now, we give the fundamental

(2.1) DEFINITION. By a smooth groupoid we mean groupoid (1.1) in which the sets $\alpha^{-1}(x)$, $x \in V$, are leaves of the d.s. Φ .

The set $\alpha^{-1}(x)$ equipped with the suitable d.man. structure is called the <u>leaf of this groupoid over x</u> and denoted by Φ_x , xeV. The mappings $D_h: \Phi_{\beta h} \rightarrow \Phi_{\gamma h}, g \mapsto g \cdot h,$ (2.2)h€Φ, are diff.'s. Every diff. groupoid is, of course, a smooth groupoid; the proper d.subman. $\alpha^{-1}(x)$ of Φ is a leaf over x. xeV. (2.3) PROPOSITION. Groupoid (1.4) is a smooth groupoid iff each abstract class of R is a leaf of V. In this case, the mapping (2.4) $\boldsymbol{\gamma}_{\mathbf{x}}: \boldsymbol{L}_{\mathbf{x}} \longrightarrow \boldsymbol{R}_{\mathbf{x}}, \ \mathbf{y} \longmapsto (\mathbf{x}, \mathbf{y}),$ is a diff. of leaves (L_x - the abstract class of R through x). (2.5) THEOREM. Let (1.1) be any Lie groupoid. Then, for an equivalence relation R for which (1.4) is a smooth groupoid, the subgroupoid $\Phi^{R} = (\alpha, \beta)^{-1} \Gamma R$ (2.6)equipped with the d.str. of a proper d.subs. of Φ , turns out to be a smooth groupoid. <u>Proof.</u> Of course, $\mathbf{\Phi}^{R}$ forms a groupoid. Let xeV. Consider the sub-<u>Proof.</u> Of course, Ψ forms a groupold. Let xev. Consider the sub-mersion $\beta_x: \Phi_x \longrightarrow V$, $h \mapsto \beta h$, and take the abstract class L_x of R th-rough x. The inverse image $\beta_x^{-1}[L_x]$ forms in a natural manner an im-merse d.subman. Φ_x^R of the d.man. Φ_x , characterized by the property: - if $A \subset L_x$ and the d.man. $L_{x|A}$ is a proper d.subman. of V, then $\beta_x^{-1}[A] \subset \Phi_x^R$ and the d.man. $\Phi_{x|B-1}^R[A]$ is a proper d.subman. of Φ_x . Of course, $\beta_x^{R}: \Phi_x^R \longrightarrow L_x$ - the induced mapping - is a submersion. We have to show that Φ_x^R is a leaf of the d.s. Φ^R , which is equivalent to the fact that Φ_x^R is a leaf of the d.s. Φ^R . fact that Φ_r^R is a leaf of the d.man. Φ_r .

Let X be any locally arcwise connected topological space and f:X $\rightarrow \Phi_x$ - any continuous mapping such that $f[X] \subset \Phi_x^R$. Take an arbitrary point $t_{\mathcal{C}}X$. By the submersivity of \mathcal{B}_x , there exist nbh's $\mathbb{W} \subset \Phi_x$ and $\mathbb{W} \subset \mathbb{V}$ of $f(t_0)$ and $y_0 := \mathcal{B}(f(t_0))$, respectively, and a diff. $\psi : \mathbb{W} \to \mathbb{W} \times \mathbb{R}^S$, $s = \operatorname{codim} L_x$, such that $\psi^1 : \mathbb{W} \to \mathbb{W}$, $h \mapsto \operatorname{pr}_1 \circ \psi(h)$, is equal to $\mathcal{B}_x \mathbb{W}$. Take any subset $\mathbb{U} \subset L_x$, containing y_0 , such that $L_{x | \mathbb{U}}$ is a proper d. subman. of V. Put $U = \psi^{-1}[\mathbb{U} \times \mathbb{R}^S]$. Of course, $U = (\mathbb{W} \cap \Phi_x^R) \cap \mathcal{B}_x^{-1}[\mathbb{U}] \subset \Phi_x^R$ and $\varphi : \Phi_{x | \mathbb{U}}^R \to L_{x | \mathbb{U}} \times \mathbb{R}^S$, $h \mapsto \psi(h)$, is a diff. Let $B = f^{-1}[\mathbb{W}]$. Then f(B) $\mathbb{C} \mathbb{W} \cap \Phi_x^R$, so the image of the mapping $\psi^1 \circ f(B)$ (which is equal to $\mathcal{B}_x \circ f(B)$) is contained in $\mathbb{W} \cap L_x$. From the assumption about R it follows that $\psi^1 \circ f(B:B \to L_x) = L_x | \mathbb{U}$ is continuous. Put $\widetilde{B} = (\psi^1 \circ f(B)^{-1}[\mathbb{U}]$. $\widetilde{B} \in \operatorname{Top} X$ and $\psi^1 \circ f(B:\widetilde{B} \to L_x | \mathbb{U}$ is continuous, so is $f(\widetilde{B} = \varphi^{-1} \circ (\psi^1 \circ f(\widetilde{B}), \psi^2 \circ f(\widetilde{B})) : \widetilde{B} \to \Phi_x^R$. The free choice of $t_0 \in X$ implies that $f:X \to \Phi_x^R$ is continuous.

3. ALGEBROID OF A SMOOTH GROUPOID. The present author's observa-

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tions show that by following the idea of J.Pradines [12],[13], one can construct an object analogous to the Lie algebroid of a diff. groupoid for groupoids from a much wider class, namely for smooth groupoids.

We first recall the notion of a tangent vector and the tangent d. s. over a d.s. By a <u>tangent vector to a d.s. (M,C) at xeM</u> [19] we mean each linear mapping v: $C \rightarrow \mathbb{R}$ such that $v(f \cdot g) = v(f) \cdot g(x) + f(x) \cdot v(g)$ for f,geC. All tangent vectors at x form a vector space which is denoted by $T_x(M,C)$ and called the <u>tangent vector space at x</u>. By the<u>differential at x</u> [20] of any smooth mapping $f:(M,C) \rightarrow (N,D)$ between d.s.'s (M,C) and (N,D) we mean the linear mapping $f_{*x}:T_x(M,C) \rightarrow$ $T_{f(x)}(N,D)$ defined by the formula $f_{*x}(v)(g) = v(g \circ f)$, geD, $v \in T_x(M,C)$. If (N',D') is a d.subs. of (N,D) and $i:(N',D') \rightarrow (N,D)$ denotes the inclusion, then, for $y \in N'$, $i_{*y}:T_y(N',D') \rightarrow T_y(N,D)$ is a monomorphism; with its help the space $T_y(N',D')$ is identified with the vector subspace Imi_{*v} of $T_v(N,D)$.

Let (M,C) be any d.s. We put

(i) $T(M,C)=\bigsqcup_{x\in M}T_x(M,C)$ (the disjoint union of all tang. spaces), (ii) $\pi:T(M,C) \longrightarrow M$ - the canonical projection,

(iii) TC= $(\operatorname{sc}^{\widehat{C}})_{T(M,C)}$ where $\widehat{C}=\{g\circ\pi; g\in C\}\cup\{dg; g\in C\}\ (dg:T(M,C) \longrightarrow \mathbb{R}, v \mapsto v(g)).$

Following A.Kowalczyk [5], the d.s. (T(M,C),TC) is called the <u>tangent d.s.</u> to a d.s. (M,C). Any (smooth) section X:M \rightarrow T(M,C) of π is called a (<u>smooth</u>) vector field on (M,C). The smoothness of X is equivalent to the fact that X(f) \in C for f \in C. We shall denote the C-module of all smooth vector fields on (M,C) by $\mathfrak{X}(M,C)$.

In the sequel of this chapter, we fix a smooth groupoid (1.1). Let C and D denote the d.str.'s of Φ and V, respectively. This determines the system

(3.1)

$(A(\Phi), p, V)$

in which (1) $A(\Phi)$ is the proper d.subs. of the tangent d.s. $T\Phi$ with the support $\bigsqcup_x T_{u_x} \Phi_x \subset T\Phi$ (we recall that Φ_x denotes the leaf of Φ over x, $x \in V$), (2) $p:A(\Phi) \longrightarrow V$ is the projection defined by p(v)=xiff $v \in T_{u_x} \Phi_x$. Of course, p is smooth. The structure of a vector space is defined in each fibre of p. Unfortunately, system (3.1) is not in general - a vector bundle (even if V is a manifold). However, it has some interesting properties described in theorem (3.7) below.

A vector field X on Φ is called an <u> α -field</u> if $X_h \in T_h \Phi_{\alpha h}$, he Φ . An α -field X is called <u>right-invariant</u> (briefly r-i) if $(D_h)_{\star g}(X_g) = X_{g \cdot h}$, $g, h \in \Phi$ and $\alpha g = \beta h$, where D_h are mappings (2.2). It is easy to see that the Lie bracket of smooth r-i vector fields is such a field, too. All smooth r-i vector fields on Φ form an R-Lie algebra (w.r.t. the Lie bracket) and a D-module (w.r.t. the multiplication f•X:=f•B•X) denoted by $\mathbf{\mathfrak{X}}^{\mathrm{R}}(\boldsymbol{\Phi})$. SecA($\boldsymbol{\Phi}$) - the vector space of all global smooth sections of p - forms a D-module, too.

Each smooth r-i vector field X determines a smooth section

$$X_{\alpha}: \mathbb{V} \longrightarrow \mathbb{A}(\Phi), \ \mathbf{x} \longmapsto X_{\mathbf{u}_{\mathbf{x}}},$$

of p. The mapping (3.2)

 $\mathfrak{X}^{R}(\Phi) \longrightarrow \operatorname{Sec} \mathbb{A}(\Phi), X \longmapsto X_{\alpha},$ is a homomorphism of D-modules. Conversely, we have the following (3.3) PROPOSITION. For each $\eta \in \text{Sec A}(\Phi)$, there exists exactly one smooth r-i vector field on Φ , denoted by η' , such that

 $\eta'_{u_x} = \eta_x$, xeV.

(*)The mapping

Sec $A(\Phi) \longrightarrow \mathfrak{X}^{R}(\Phi), \eta \mapsto \eta',$ (3.4)

is an isomorphism of D-modules, inverse to (3.2).

<u>Proof</u>. Let $\eta \in \text{Sec A}(\Phi)$. A r-i vector field η' on Φ , such that (*)holds, is defined by the formula

(**) $\eta'_{h} = (D_{h})_{*u_{\beta h}}(\eta_{\beta h}), h \in \Phi,$ which proves the uniqueness. To show the existence, we must prove that the vector field η' defined by (**) is r-i and smooth. This first fact is easy to see. To prove the second, we take an arbitrary f eC. For he $\overline{\Phi}$, we have $\eta'_{h}(f) = \eta \circ \beta(h)(\overline{\Phi}_{\beta h}) \Rightarrow f^{\circ}(\cdot)(g,h)$. From the assumption that $\cdot: \Phi * \Phi \longrightarrow \Phi$ is smooth we have $f \circ (\cdot) \in (C \times C)_{\Phi * \Phi}$. We fix $h_{c} \in \Phi$ and find a nbh $\Omega \in \mathcal{T}_{CRC}$ of $(u_{Bh_{c}}, h_{c})$ and a function fecar such that $f \circ (\cdot) I \Omega \cap (\Phi * \Phi) = \tilde{f} I \Omega \cap (\Phi * \Phi)$. Thus, for h from some nbh of h, we have $\eta'_{h}(f) = \eta \circ \beta(h)(\tilde{f}(\cdot,h))$. The function $\Phi \to H \mapsto Y(h)(\tilde{f}(\cdot,h))$, where $Y: \overline{\Phi} \to T\overline{\Phi}$, $g \mapsto \eta \circ \beta(g)$, belongs to C; see the lemma below. (3.5) LEMMA. Let (M, \tilde{C}) and (N, \tilde{D}) be any d.s.'s and Y: $(N, \tilde{D}) \rightarrow$ $(T(M, \tilde{C}), T\tilde{C})$ - any smooth mapping. Then, for any smooth function \tilde{f} $\epsilon \tilde{c} \mathbf{X} \tilde{D}$, the following function N $\ni x \mapsto Y(x)(\tilde{f}(\cdot, x)) \epsilon \mathbb{R}$ belongs to \tilde{D} .

<u>Proof</u>. Put $Y_1: \mathbb{N} \to T(\mathbb{M} \times \mathbb{N}), x \mapsto (Y(x), \theta_x), \text{ where } \theta_x \in T_x \mathbb{N} \text{ denotes}$ the null vector. We prove that $Y_1:(N,\tilde{D}) \rightarrow (T(M\times N),T(\tilde{C}\times \tilde{D}))$ is smooth. $\tilde{C}\times\tilde{D}$ is generated by $E = \{g \circ pr_1; g \in \tilde{C}\} \cup \{h \circ pr_2; h \in \tilde{D}\}, so [5] T(\tilde{C}\times\tilde{D})$ is generated by $\tilde{E} = \{\delta \circ \Pi; \delta \in E\} \cup \{d\delta; \delta \in E\}$. To see the smoothness of Y_1 , we have to notice only that $\zeta \cdot Y_1 \in \widetilde{D}$ for $\zeta \in \widetilde{E}$. In the end, we see that $(x \mapsto Y(x)(\tilde{f}(\cdot, x))) = d\tilde{f} \circ Y_1 \in \tilde{D}.$

(3.6) DEFINITION. By a vector pseudobundle (over a d.s. V) we mean each system (A,p,V) containing d.s.'s A and V and a surjective smooth mapping p:A \rightarrow V in whose fibres some structures of vector spaces are defined and the following properties hold:

 $(1) + : A \oplus A \longrightarrow A, (v, w) \longmapsto v + w, : : \mathbb{R} \times A \longrightarrow A, (r, v) \longmapsto r \cdot v, are smo-$

oth mappings where $A \oplus A$ denotes the proper d.subs. of $A \times A$ with the support {(v,w) ext{A}; pv=pw},

(2) for any number meN, any smooth sections ξ_1, \ldots, ξ_m of p and any set $U \subset V$ (necessarily open) such that the vectors $\xi_1(x), \ldots, \xi_m(x)$ are linearly independent for $x \in U$, the mapping

 $\varphi: \mathbb{V}_{\mathrm{III}} \times \mathbb{R}^{\mathrm{m}} \to \mathbb{A}, (\mathbf{x}, \mathbf{a}) \mapsto \sum_{i} \mathbf{a}^{i} \boldsymbol{\xi}_{i}(\mathbf{x}),$ is a diff. onto its image.

Let (A, p, V) and (A', p', V') be two vector pseudobundles. By a homomorphism between them we mean a pair of smooth mappings (h,H), h:V+V, H:A \rightarrow A', such that p'•H=h•p and for each xeV, H₁:A₁ \rightarrow A'₁h(x) is a linear homomorphism of vector spaces. If V=V' and h=id, then this homomorphism is called strong and denoted by one letter H. All vector pseudobundles and homomorphisms form a category. (3.7) THEOREM. System (3.1) is a vector pseudobundle.

<u>Proof</u>. Extending sections $\xi_1, \ldots, \xi_m \in \text{Sec A}(\Phi)$ to smooth vector fields on Φ (for example, to r-i vector fields), we see that the theorem is an immediate consequence of the following lemma. (3.8) LEMMA. The tangent d.s. $T(M,\tilde{C})$ to a d.s. (M,\tilde{C}) is a vector pseudobundle.

Proof. The smoothness of + and • is easy to see. Now, take meN, any smooth vector fields $X_1, \ldots, X_m \in \mathfrak{X}(M, \tilde{C})$ and any subset $U \subset M$ (not necessarily open) such that the vectors $X_1(x), \ldots, X_m(x)$ are linearly independent for each xeU and define $\varphi: \mathbb{M}_{UV} \times \mathbb{R}^m \longrightarrow \mathbb{T}(\mathbb{M}, \mathbb{C})$, $(x, a) \mapsto$ $\sum_{i} a^{i} X_{i}(x)$. The smoothness of φ is evident. To prove this for φ^{-1} , we put $\operatorname{pr}_1: U \times \mathbb{R}^m \longrightarrow U$, $(x, a) \longmapsto x$, and $\operatorname{p}^s: U \times \mathbb{R}^m \longrightarrow \mathbb{R}$, $(x, a^1, \ldots, a^m) \longmapsto$ a^s, $s \leq m$. Of course $pr_1 \circ q^{-1} = \pi i q [U \times \mathbb{R}^m]$ is smooth. So, it suffices to a⁵, s<m. Of course pr₁• φ = $\pi_{I}q_{IUXR^{-1}}$ is smooth. So, it suffices to show that p⁸• $\varphi^{-1} \in (T\tilde{C})_{\varphi[UXR^{m}]}$. For this purpose, we notice that p⁸• $\varphi^{-1}(\sum_{i}a^{i}X_{i}(x))=a^{8}$, and we take any point $x_{o}\in U$ and functions f¹, ..., f^m \in C such that $X_{i}(f^{j})(x_{o})=\delta_{i}^{j}$ [21]. Then, for some nbh \tilde{U} of x_{o} , we have: det $[X_{i}(f^{j})(x)] \neq 0$, $x\in \tilde{U}$, and we can define the mapping $\psi:\tilde{U} \rightarrow GI(m,R), x \mapsto [X_{i}(f^{j})(x)]$. Let $^{-1}\circ\psi(x)=[c_{j}^{k}(x); j,k\leqslant m], x\in \tilde{U}$. For $F:R^{2m} \rightarrow R$, $(x^{1},...,x^{m},y^{1},...,y^{m}) \mapsto \sum_{j} y_{j}x_{j}$, we have (1) $F(c_{1}^{8}\circ\pi,...,c_{m}^{8}\circ\pi,df^{1},...,df^{m})\in(T\tilde{C})_{\pi}-1[\tilde{U}],$ (2) $F(c_{1}^{8}\circ\pi,...,c_{m}^{8}\circ\pi,df^{1},...,df^{m})\eta\pi^{-1}[\tilde{U}] \cap \varphi[U\times R^{m}]$ $=r^{8}\circ m^{-1}m^{-1}UU \cap gUU\times R^{m}$ which ends the proof.

 $= p^{s} \circ \varphi^{-1} | \pi^{-1} [\tilde{U}] \cap \varphi [U \times \mathbb{R}^{m}] \text{ which ends the proof.} \blacksquare$

For $\xi, \eta \in \text{Sec A}(\Phi)$, we put

(3.9) $[[\xi,\eta]] := [\xi,\eta]_{0},$ and take the mapping $\tilde{B}_*: A(\Phi) \longrightarrow TV$, $v \longmapsto B_*v$. For any $\xi \in Sec \Lambda(\Phi)$, the r-i vector field ξ on Φ is B-related to the vector field $\tilde{B}_{\mathfrak{s}} \mathfrak{e} \mathfrak{I} \mathfrak{V}$) (3.10) DEFINITION. The system $(A(\overline{\Phi}), [\cdot, \cdot], \widetilde{\beta}_{+})$ (3.11)

is called the algebroid of the smooth groupoid §.

The fundamental properties of system (3.11) are described by

(3.12) THEOREM. (a) The system (Sec $A(\Phi)$, $[\cdot, \cdot]$) is an R-Lie algebra, (b) the mapping Sec \tilde{B}_{*} :Sec $A(\Phi) \rightarrow \mathfrak{X}(V)$, $\xi \mapsto \tilde{B}_{*} \circ \xi$, is a homomorphism of Lie algebras,

(c) the equality $\llbracket \xi, f \cdot \eta \rrbracket = f \cdot \llbracket \xi, \eta \rrbracket + (\tilde{B}_* \circ \xi)(f) \cdot \eta$ holds for $\xi, \eta \in Sec A(\Phi)$ and $f \in D$.

<u>Proof.</u> (a) is easy to see, (b) Let $\xi,\eta\in\operatorname{Sec} A(\Phi)$. Since ξ' is B-related to $\tilde{B}_{*}\circ\xi$ and $\eta' - to \tilde{B}_{*}\circ\eta$, therefore $\lfloor\xi,\eta'\rfloor$ is B-related to $[\tilde{B}_{*}\circ\xi,\tilde{B}_{*}\circ\eta]$. Also, $\llbracket\xi,\eta \rfloor'$ is B-related to $\tilde{B}_{*}\circ \llbracket\xi,\eta \rfloor$, thus by (3.9) and by the surjectivity of B, we see that $\tilde{B}_{*}\circ \llbracket\xi,\eta \rfloor = [\tilde{B}_{*}\circ\xi,\tilde{B}_{*}\circ\eta]$. (c) follows from the equality $\llbracketX,f\circ B\cdot Y\rrbracket = f\circ B\cdot \llbracketX,Y\rrbracket + (\tilde{B}_{*}\circ X_{O})(f)\circ B\cdot Y$ for $X,Y\in\mathfrak{X}^{R}(\Phi)$ and feD.

The properties of system (3.11) described in theorems (3.7) and (3.12) suggest the following definition of an abstract algebroid. (3.13) DEFINITION. By an <u>algebroid</u> we mean a system (3.14) (A, **L**•, •**D**, γ),

consisting of (1) a vector pseudobundle A = (A, p, V), (2) a mapping **[**·, ·]]:Sec AXSec A \rightarrow Sec A where Sec A denotes the D-module of all smooth sections of p (D - the d.str. of V), (3) a mapping $\gamma: A \rightarrow TV$, such that (a) the system (Sec A, [·, ·]) is an R-Lie algebra, (b) the mapping Sec $\gamma:$ Sec A $\rightarrow \mathcal{X}(V)$, $\xi \mapsto \gamma \circ \xi$, is a homomorphism of Lie algebras, (c) **[** ξ , f· η] = f·**[** ξ , η] + ($\gamma \circ \xi$)(f)· η for ξ , $\eta \in$ Sec A, f \in D.

If A is a vector bundle (over a manifold V), then (3.14) is simply a <u>Lie algebroid</u> in the sense of J.Pradines [12], [13].

For two algebroids $(A, \mathbf{I}, \mathbf{J}, \gamma)$ and $(A', \mathbf{I}, \mathbf{J}', \gamma')$ over the same d.s. V, by a (strong) homomorphism between them we mean a homomorphism of vector pseudobundles H:A $\rightarrow A'$, such that $(1)\gamma' H=\gamma$, (2) Sec H: Sec A \rightarrow Sec A' is a homomorphism of R-Lie algebras.

Any strong homomorphism $F: \Phi \to \Phi'$ between two smooth groupoids determines a homomorphism of their algebroids $\widetilde{F}_{\#}: A(\Phi) \to A(\Phi')$, $v \mapsto F_{\#}v$. The covariant functor $\Phi \mapsto A(\Phi)$, $F \mapsto \widetilde{F}_{\#}$, obtained above is called (like in J.Pradines [13] for diff. groupoids) the <u>Lie functor</u>.

It is possible that definitions (3.6) and (3.13) are too general. The answer to the following question will solve this problem. (3.15) THE FIRST OPEN PROBLEM. Is each algebroid isomorphic to the algebroid of some smooth groupoid? If the answer is negative, find any necessary and sufficient conditions for the algebroid to be isomorphic.

4. FRADINES-TYPE GROUPOIDS.

(4.1) DEFINITION. By a groupoid of Pradines type we shall mean any smooth groupoid (1.1) in which V is a d.man. and system (3.1) is a vector bundle.

(4.2) EXAMPLES. (1) Differential groupoid (1.1) is of Pradines type. Indeed, $\Lambda(\Phi) \cong u^* T^{\alpha} \Phi$ where $T^{\alpha} \Phi = \operatorname{Ker}_{\pi}$.

(2) The groupoid of the equivalence relation R determined by a foliation F of a d.man. V is of Pradines type. Indeed, $A(R) \cong TF$.

(3) The smooth groupoid Φ^{R} , defined in theorem (2.5) with the help of an equivalence relation R for which the family of abstract classes is a foliation \mathfrak{F} of V, is a groupoid of Pradines type. In fact, $A(\Phi^{R}) = \tilde{B}_{*}^{-1}$ [TF] is a vector subbundle of $A(\Phi)$.

By the remark following definition (3.13), we obtain (4.3) COROLLARY. If (11) is a groupoid of Pradines type, then system (3.11) is a Lie algebroid.

(4.4) THEOREM. Smooth groupoid (1.1) in which V is a d.man. is a groupoid of Pradines type iff

(i) for each vector $v \in A(\Phi)$, there exists $\xi \in Sec A(\Phi)$ such that $\xi(pv) = y$ (ii) the function $Vax \longrightarrow \dim A(\Phi)$ is constant.

(ii) the function $\forall \exists x \mapsto \dim A(\Phi)_{|x}$ is constant. <u>Proof</u>. " \Rightarrow " is evident, " \notin ". Let $x \in V$ and let (v_1, \dots, v_m) be any basis of $A(\Phi)_{|x}$. Take $\xi_1, \dots, \xi_m \in \text{Sec } A(\Phi)$ and a nbh U of x in V such that $\xi_1(x) = v_1$, $i \in m$, and the vectors $\xi_1(y), \dots, \xi_m(y)$ are linearly independent for $y \in U$. We need notice that

 $\varphi: \mathbb{U} \mathbb{R}^m \longrightarrow p^{-1}[U], (y, a^1, \dots, a^m) \longmapsto \sum_i a^i \xi_i(y)$ is a diffeomorphism. This is an consequence of theorem (3.7).

Now, let R be any equivalence relation on a d.man. V. In the theorem below, we give the complete answer to the question when the groupoid of R is of Pradines type. We see that - because of statement (2) - this theorem may be considered as the <u>next generalization of</u> <u>Godement's theorem on division</u> (see [10]).

(4.5) THEOREM. The following conditions are equivalent:

(1) (1.4) is a groupoid of Pradines type with $\dim \mathbb{R}_{x} = k$, $x \in \mathbb{V}$.

(2) The family \mathcal{F} of all arcwise connected components of all abstract classes of R is a k-dim. foliation.

(3) (a) (1.4) is a smooth groupoid, and

(b) there exists a subset $\Omega \subset \mathbb{R}$ such that

(i) $\Delta \subset \Omega$ where $\Delta = \{(x, x) \in \mathbb{V} \times \mathbb{V}; x \in \mathbb{V}\},\$

(ii) Ω is a proper n+k-dim. d.subman. of V×V,

(iii) $\operatorname{pr}_1 I \Omega : \Omega \rightarrow V$ is a submersion,

(iv) for each xeV, the proper d.subman.

$$\Omega_{\mathbf{x}} := (\mathrm{pr}_1 | \Omega)^{-} (\mathbf{x})$$

of Ω is an open d.subman.of the leaf R.

(4) There exists a subset $\Omega \subset \mathbb{R}$ such that (i); (iii) as above.

- (iv) $D_{(x,y)}[\Omega_y] \cap \Omega_x \sigma \Omega_x$ for $(x,y) \in \mathbb{R}$,
- (v') connected components of the manifold \tilde{R}_x , xeV, (see lemma in [10]) are equal to arcwise connected components of the set R_x in V*V.

<u>Proof.</u> (1) \neq (2). Let us take any abstract class L of R and xeL. We define on L a d.str. of a d.man. in such a way that the mapping γ_x : L $\rightarrow R_x$, $y \mapsto (x, y)$, is a diffeomorphism. L is an immerse subman. of V.From the fact that R_x is a leaf of V×V we see that connected components of the manifold L are equal to arcwise connected components of the set L in V, so the connected compone.t B_x of x of the manifold L is an element of \mathcal{F} , and $T_x B_x^{=} (pr_2)_{*(x,x)} [A(R)_{1x}] = (pr_2 IR)_{*x} [A(R)_{1x}]$. Since $(pr_2 IR)_{*} (A(R))_{*}$ is a vector subbundle of TV. Thus (2) results now from some version of Frobenius' theorem [1, p.86].

(2) \Rightarrow (3). Condition (a) follows from the observation that each abstract class of R is a leaf of V (w.r.t. the definition contained in chapter 2). To prove (b), we take any nice covering $\{(U_i, \varphi_i, \mathbb{R}^n); i \in \mathbb{N}\}$ of \mathcal{F} , n=dim V. Let us denote by Q_x^i the plaque of the chart (U_i, φ_i) which contains x, $x \in U_i$. Like in the proof of theorem 2 from paper [10], we see that $\Omega = \bigcup_i \Omega_i$, where $\Omega_i = \{(x, y) \in \mathbb{V} \times \mathbb{V}; x \in U_i, y \in Q_x^i\}$, has properties (i): (iii). To show (iv), we notice that the inclusion $\Omega_x \hookrightarrow \mathbb{R}_x$ is an immersion, and that dim $\Omega_x = \dim \mathbb{R}_x$.

 $(3) \Rightarrow (4)$. Let Ω fulfil (i); (iv). Condition (iv) holds in an evident manner. To show (v), it is sufficient to notice that the manifold \tilde{R}_{τ} is equal to the leaf R_{τ} , xeV.

 $\underbrace{(4) \neq (1)}_{X} \text{ Let us assume that a subset } \Omega \subset \mathbb{R} \text{ has properties (i)}_{:}$ (iii), (iv) and (v). Take any abstract class L of R and xeL. Via bijection $\gamma_{X}: L \to \widetilde{R}_{X}, y \mapsto (x, y)$, we define some d.str. of a d.man. on L such that γ_{X} is a diff. By property (i) of \widetilde{R}_{X} (see lemma in [10]), we see the correctness of the definition of the d.man. L, while by property (ii) of \widetilde{R}_{X} - that the inclusion $L \hookrightarrow V = (L \xrightarrow{\gamma_{X}} \widetilde{R}_{X} \hookrightarrow \{x\} \times V \xrightarrow{S} V)$ is an immersion. In view of assumption (v), the family \mathfrak{F} of all arcwise connected components of all abstract classes of R is equal to the family of all connected components of all manifolds L obtained above. Let $x \in B \in \mathfrak{F}$. From the definition of \widetilde{R}_{X} we see that $\Omega_{X} \subset \widetilde{R}_{X}$, thus $(4.6) \qquad T_{Y} \mathbb{B} = (pr_{2})_{*} (x, y) \Omega_{Y} \mathbb{I}$.

(4.6) $T_x^{B=}(pr_2)_{*}(x,x)^{LT}(x,x)\Omega_x^{J}$. Put $T^{\alpha}\Omega := \text{Ker } \alpha_*, \alpha := pr_1 \mid \Omega$. It is a vector subbundle of order k of the tangent bundle $T\Omega$, which implies that $\hat{u}^*T^{\alpha}\Omega$ ($\hat{u}: V \to \Omega$, $x \mapsto (x,x)$) is a vector bundle of order k over V. Since $x:\hat{u}^*T^{\alpha}\Omega \to TV$, $v \mapsto (pr_2)_* v$, is a monomorphism of vector bundles (see the proof of

theorem 2 in [10], part $(2) \Rightarrow (1)$, therefore, by (4.6), we see that TF is a vector subbundle of TV, which implies that F is a k-dim. foliation and, next, that each element $B \in \mathfrak{F}$ is a leaf of V. Thus each d.man. L obtained above is a leaf, too, which gives that $\tilde{R}_{_{\mathbf{Y}}}$ is aleaf of groupoid (1.4). Then, this last is a smooth groupoid. We have $A(R)_{|x} = T_{(x,x)} \tilde{R}_{x} = (T^{\alpha} \Omega)_{|(x,x)} \subset TR.$ To finish the proof, we need notice the equality of d.s.'s $A(R) = T^{\alpha} \Omega_{|\Delta}$. Since A(R) is (by definition) tion) a proper d.subs. of TR and $T^{\alpha}\Omega_{1\Lambda}$ - of T Ω , it suffices to show that T Ω is a proper d.subs. of TR, but this results from the following lemma used to the situation when $\Omega \hookrightarrow R_{\bullet}$ (4,7) LEMMA. If (M',C') is a d.subs. of (M,C), then the tangent d.s. (T(M',C'),TC') is a d.subs. of the tangent d.s. (T(M,C),TC), what is more, if for $U \in \tau_{C'}$, the equality $C'_{U} = C_{U}$ holds, then $(TC')_{\pi'}$ =(TC)_{π' -1_{IIII} where $\pi': T(M', C') \rightarrow M'$ is the projection. In particular,} if (M',C') is a proper d.subs., then (T(M',C'),TC') is such, too.

Theorem (4.5) is the source of the notion of a nice structure (recall that the existence of the set Ω in that theorem is proved by using a nice covering of a foliation).

(4.8) DEFINITION. By a k-dim. nice structure of groupoid (1.1) we shall mean any subset $\Omega \subset \Phi$ such that

(i) u[V] $\subset \Omega$,

(ii) Ω is an n+k-dim. proper d.subman. of Φ ,

(iii) $\alpha \mid \Omega : \Omega \rightarrow V$ is a submersion.

(iv) $D_h[\Omega_y] \cap \Omega_x \sigma \Omega_x := (\alpha | \Omega)^{-1}(x), h \in \Phi, x = \alpha h, y = \beta h,$

(v) connected components of the d.man. $\tilde{\Phi}_{v}$ (see lemma below) are equal to arcwise connected components of the subset $\propto^{-1}(x)$ in Φ . (4.9) LEMMA. If $\Omega \subset \Phi$ has properties (i) ÷ (iv) above, then, for each point $x \in V$, there exists exactly one k-dim. C^{OD}-manifold $\tilde{\mathbf{F}}_{\mathbf{x}}$ with the support $\alpha^{-1}(\mathbf{x})$, such that, for each $h \in \alpha^{-1}(\mathbf{x})$,

(a) $D_h[\Omega_v] \sigma \tilde{\Phi}_v$,

(b) $D_h I \Omega_y : \Omega_y \rightarrow D_h [\Omega_y] \sigma \tilde{\Phi}_x$ is a diff. The manifolds $\tilde{\Phi}_x$ have the properties: (i) $D_h : \tilde{\Phi}_{\beta h} \rightarrow \tilde{\Phi}_{\alpha h}$ is a diff., (ii) $\tilde{\Phi}_x$ are Hausdorff, (iii) $\tilde{\Phi}_x$ are d.subs. of Φ , (iv) if, in addition, this groupoid is a smooth groupoid with dim $\Phi_x = k [=k']$ and the man. $\tilde{\Phi}_{\mathbf{x}}$ has a countable basis], then [k=k' and] the leaf $\Phi_{\mathbf{x}}$ is equal to $\tilde{\phi}_x$, xeV.

The proof is similar to that of the lemma from [10]. We only prove properties (ii) and (iv). To prove (ii), we first notice the continuity of the inclusion $j: \tilde{\Phi}_x \to \Phi$ in some nbh of any point $h \epsilon \alpha^{-1}(x)$: $j: \tilde{\varPhi}_{\mathbf{x}} \mathfrak{D}_{\mathbf{h}}[\Omega_{\mathbf{\beta}\mathbf{h}}] \xrightarrow{\mathbb{D}_{\mathbf{h}}^{-1}} \hat{\Omega}_{\mathbf{\beta}\mathbf{h}} \xrightarrow{\mathbb{C}} (\varPhi_{\mathbf{\beta}\mathbf{h}}, \mathbb{C}_{\varPhi_{\mathbf{\beta}\mathbf{h}}}) \xrightarrow{\mathbb{D}_{\mathbf{h}}} (\varPhi, \mathbb{C}).$

Next, (ii) follows from the fact that (Φ, τ_c) is Hausdorff. To prove

(iv), we notice that the identity map $\tilde{\Phi}_x \rightarrow \Phi_x$ is an immerse bijection between manifolds of the same dimensions [4,p. 101]. (4.10) THE SECOND OPEN PROBLEM. Are the manifolds $\tilde{\Phi}_x$ leaves of Φ ? (4.11) DEFINITION. By a nice structure of smooth groupoid (1.1) we ahall mean any nice structure of this groupoid for which $\tilde{\Phi}_x = \Phi_x$, xeV.

We see that $\Omega \subset \Phi$ is a nice structure of smooth groupoid (1.1) if (i)÷(iii) as in (4.8), (iv') Ω_{χ} is an open subman. of Φ_{χ} .

The notion of a groupoid with a nice structure is closely connected with the notion of "un morceau differentiable de groupoide" in the sense of J.Pradines [11].

(4.12) THEOREM. A smooth groupoid which has a nice structure is of Pradines type.

Proof. Let Ω be a nice structure of smooth groupoid (1.1), k= =dim Φ_x . Then $T^{\alpha}\Omega$:=Ker($\alpha | \Omega \rangle_*$ is a vector subbundle of $T\Omega$ of order k. Thus $T^{\alpha}_{o}\Omega$:= $T^{\alpha}\Omega_{|u|}(V)$ is a vector bundle of order k over u[V]. Next, we see that $A(\Phi)_{|x} = T_{u_x}(\Phi_x) = T_{u_x}(\Omega_x) = (T^{\alpha}_{o}\Omega)_{|u_x}$, and that, by (4.7), $T^{\alpha}\Omega$ is a proper d.subs. of $T\Phi$. Thus $A(\Phi) = T^{\alpha}\Omega_{|u|}(V)$ as d.s.'s, which implies that $A(\Phi)$ is a vector bundle.

(4.13) THE THIRD OPEN PROBLEM. Has every Pradines-type groupoid a nice structure?

(4.14) REMARK. Let a family \mathfrak{F} of immerse connected submanifolds of V (covering V) be given. Take the groupoid (1.4) of the equivalence relation R whose family of abstract classes is equal to \mathfrak{F} . We see that theorem (4.5) can be formulated as follows:

The conditions are equivalent:

- (1) R is of Pradines type,
- (some version of Frobenius' theorem [1,p.86])
- (2) F is a foliation,
 - \$ (some generalization of Godement's theorem [10])
- (3) R has a nice structure.

(4.15) EXAMPLE. Let G be a non-connected Lie group. G is, of course, a Lie groupoid with the one-point manifold of units. Each (open) nbh U of the neutral element eeG is a nice structure of this groupoid. It is well known that each connected nbh U \ni e generates only the connected component of the element e in G.

Let Ω be any subset of algebraic groupoid (1.1). By the <u>groupoid</u> <u>generated by Ω </u> we mean the smallest algebraic subgroupoid Ψ of Φ containing Ω . It is easy to see that Ψ consists of all finite products $h_n \cdot \ldots \cdot h_1$ only, where $h_1 \epsilon \Omega \cup \Omega^{-1}$ ($\Omega^{-1} = \{h^{-1}; h \epsilon \Omega\}$), $i \leq n$, $n \epsilon N$. (4.16) <u>PROPOSITION</u>. If Ω is a nice structure of groupoid (1.1) and Ψ is the subgroupoid generated by Ω , then the set $\Psi_r := \Psi \cap \alpha^{-1}(x)$ is an open-closed subset of $\tilde{\Phi}_x$, xeV.

The very simply proof of this proposition is omitted. \blacksquare (4.17) CPROLLARY. Let F be any foliation of a d.man. V. Then each nice structure of \Im , i.e. each nice structure of Pradines-type groupoid (1.4) of the equivalence relation determined by \Im , generates the entire groupoid. \blacksquare

(4.18) DEFINITION. By a nice groupoid we shall mean a smooth groupoid Φ for which there exist nice structures Ω of Φ and Ω_0 of the groupoid of the equivalence relation

 $\begin{array}{ll} (4.19) & R_{\underline{\Phi}} = \{(x,y) \in \mathbb{V} \times \mathbb{V} : \bigvee_{h \in \underline{\Phi}} (\alpha h = x, \ \mathbb{G} h = y)\} \subset \mathbb{V} \times \mathbb{V}, \\ \text{respectively, such that the mapping} \\ (4.20) & (\alpha, \mathbb{B}) ! \Omega : \Omega \to \Omega_{\alpha} \end{array}$

is a submersion.

For a nice groupoid Φ , the mapping $\overline{B}_{\chi}: \Phi_{\chi} \to (\mathbb{R}_{\Phi})_{\chi}$, $h \mapsto (\chi, \beta h)$, is a submersion of leaves, $\chi \in V$.

The next theorem gives a class of nice groupoids. We need for this another notion from the theory of d.s.'s.

Let (M,C) and (N,D) be any d.s.'s. Smooth surjective mapping $f:(M,C) \rightarrow (N,D)$ is called <u>strong coregular</u> if for some natural number n the following property holds:

- for each point xéM there exist nbh's $U \in \tau_C$ and $W \in \tau_D$ of x and f(x), respectively, and a diff. $\psi : (U, C_U) \rightarrow (W, D_W) \times (\mathbb{R}^n, C^{0}(\mathbb{R}^n))$ such that $f | U = pr_1 \circ \psi$.

(4.21) DEFINITION. Groupoid (1.1) is called strong coregular if (4.22) $(\alpha,\beta): \Phi \longrightarrow R_{\delta}$

is a strong coregular mapping (R_{Φ} - the proper d.subs. of VxV). (4.23) EXAMPLE. Groupoid (2.6) is strong coregular. Indeed, (α , β): $\Phi \rightarrow VxV$ is strong coregular because it is coregular between d.man.' s. The strong coregularity of Φ^{R} follows now from

(4.24) Lemma. If (M,C) and (N,D) are d.s.'s and $f:(M,C) \rightarrow (N,D)$ is a strong coregular mapping then for each subset N'C N the mapping $f:M': (M', C_{M'}) \rightarrow (N', D_{N'})$, where $M'=f^{-1}(N')$, is strong coregular, too. what is more, if $(N', D_{N'})$ is a d.man., then $(M', C_{M'})$ is a d.man., too. (4.25) THEOREM. Each strong coregular groupoid Φ for which the groupoid R_{Φ} is of Fradines type is a nice groupoid (in particular, is of Pradines type).

<u>Proof.</u> Let Ω_0 be any nice structure of $\mathbb{R}_{\frac{1}{2}}$. Put $\Omega := (\alpha, \beta)^{-1} [\Omega_0]$. By lemma (4.24), Ω is a proper d.subman. of $\frac{1}{2}$ and mapping (4.20) is a submersion. Now, we show that Ω has properties (i)÷(iv) from definition (4.8) for k=dim Ω - dim V. (i) and (ii) hold in an evident manner. (iii) results from the equality $\alpha | \Omega = \operatorname{pr}_1^{\circ}(\alpha, \beta) | \Omega$. To show (iv), take an arbitrary héd and put x=ah, y=Bh. From the assumption That Ω_0 is a nice structure of \mathbb{R}_{Φ} we have $\mathbb{D}_{(x,y)}[\Omega_{0y}] \cap \Omega_{0x} \sigma \Omega_{0x}$. Now, (iv) follows from the continuity of $(\alpha, \beta) | \Omega_x : \Omega_x \to \Omega_{0x}$ and the equality $\mathbb{D}_h[\Omega_y] \cap \Omega_x = ((\alpha, \beta) | \Omega_x)^{-1} \mathbb{D}_{(x,y)}[\Omega_{0y}] \cap \Omega_{0x}]$. Lemma (4.9) states that, for each point xeV, there exists exactly one k-dim. d.man. $\tilde{\Phi}_x$ with the support $\alpha^{-1}(x)$, such that (a) and (b) from that lemma hold. Finally, it is sufficient to show that $\tilde{\Phi}_x$ is a leaf of Φ for each xeV. First, we notice that the mapping $\tilde{\mathbb{B}}_x: \tilde{\Phi}_x \to (\mathbb{R}_{\Phi})_x$, $h \mapsto (x,\beta h)$, has the property:

- for each point $(x,y) \in (\mathbb{R}_{\Phi})_{x}$, there exists a nbh Wor $(\mathbb{R}_{\Phi})_{x}$ of (x,y)such that (a) $(\mathbb{R}_{\Phi})_{x|W}$ is a proper d.subman. of the d.s. \mathbb{R}_{Φ} , (b) $\Psi_{x|B_{\Phi}^{-1}[W]}$ is a proper d.subman. of the d.s. Φ .

Indeed, for $(x,y)\in(\mathbb{R}_{\frac{1}{2}})_x$, we can put $\mathbb{W}:\mathbb{D}_{(x,y)}[\Omega_{oy}]$. Now, the theorem follows from the lemma below.

(4.26) LEMMA. Let (M,C) and (N,D) be any d.s.'s. If $g:(M,C) \rightarrow (N,D)$ is a strong coregular mapping and (M',C') and (N',D') are d.subs.'s of (M,C) and (N,D), respectively, such that (1) (N',D') is a leaf of (N,D), (2) M'=f⁻¹[N'], (3) for each point xeM', there exists a nbh $U \in \tau_{D'}$ of g(x) such that $D_U = D'_U$, $g^{-1}[U] \in \tau_{C'}$ and $C_{g^{-1}[U]} = C'_{g^{-1}[U]}$, then (M',C') is a leaf of (M,C).

The <u>proof</u> is identical with that for the analogous fact proved in theorem (2.5). \blacksquare

From this theorem we see that the strong coregular groupoid Φ^{R} from example (4.2)(3) is a nice groupoid.

Now, we explain the notion of the strong coregularity of smooth groupoids in the domain of differential groupoids.

(4.27) PROPOSITION. If (1.1) is a strong coregular differential groupoid with the connected space Φ , then equivalence relation (4.19) is regular in the sense of Godement **L**18, Ch.III.§12].

<u>Proof.</u> One should prove that (a) \mathbb{R}_{Φ} is a proper d.subman. of $V \times V$, (b) $\operatorname{pr}_1: \mathbb{R}_{\Phi} \longrightarrow V$ is a submersion. We see that (a) results from the lemma below and the assumption of the strong coregularity of (4.22), while (b) - from the equality $\alpha = \operatorname{pr}_1 \circ (\alpha, \beta)$.

The following lemma comes from the work by W.Waliszewski [24]. Now, we give a new short proof of this fact.

(4.28) LEMMA. If (M,C) and (N,D) are connected d.s.'s and (M,C) \times (N,D) is a d.man., then (M,C) and (N,D) are d.man.'s, as well.

<u>Proof</u>. Take $x_0 \in M$ and $y_0 \in N$ and put $m := \dim T_{x_0}(M, C)$, $n := \dim T_{y_0}(N, D)$. Of course, $k := m+n = \dim(M \times N, C \times D)$. There exist some nbh's $U \in T_C$ and $W \in T_D$ of x_0 and y_0 , respectively, and \mathfrak{s} diff. $\varphi : (U \times W, C_U \times D_W) \longrightarrow (\Omega, C^{\mathfrak{O}}(\mathbb{R}^K)_{\Omega})$ for some open subset $\Omega \subset \mathbb{R}^k$. We put $U_1 := q[U \times \{y_0\}]$ and $W_1 := \varphi[\{x_n\} \times W]$ and take the diff.'s

$$\begin{split} & \varphi_1 = \varphi(\cdot, y_0) : (\mathbb{U}, \mathbb{C}_{\mathbb{U}}) \longrightarrow (\mathbb{U}_1, \mathbb{C}^{\mathbf{0}}(\mathbb{R}^k)_{\mathbb{U}_1}), \\ & \varphi_2 = \varphi(\mathbf{x}_0, \cdot) : (\mathbb{W}, \mathbb{D}_{\mathbb{W}}) \longrightarrow (\mathbb{W}_1, \mathbb{C}^{\mathbf{0}}(\mathbb{R}^k)_{\mathbb{W}_1}). \end{split}$$

From the main theorem of paper [6] we infer - diminishing U and W, if necessary - that there exist some diff.'s

$$\begin{split} \psi_1 &: (\mathfrak{U}_1, \mathfrak{C}^{\mathbf{0}}(\mathbb{R}^k)_{\mathfrak{U}_1}) \to (\mathfrak{Q}_1, \mathfrak{C}^{\mathbf{0}}(\mathbb{R}^m)_{\mathfrak{Q}_1}), \ \mathfrak{Q}_1 \subset \mathbb{R}^m, \\ \psi_2 &: (\mathfrak{W}_1, \mathfrak{C}^{\mathbf{0}}(\mathbb{R}^k)_{\mathfrak{W}_1}) \to (\mathfrak{Q}_2, \mathfrak{C}^{\mathbf{0}}(\mathbb{R}^n)_{\mathfrak{Q}_2}), \ \mathfrak{Q}_2 \subset \mathbb{R}^n. \end{split}$$

Hence the superposition

($\psi_1 \circ \psi_2$) $\circ (\varphi_1 \times \varphi_2) \circ \varphi^{-1} : (\Omega, C^{\mathbf{0}}(\mathbb{R}^k)_{\Omega}) \rightarrow (\Omega_1 \times \Omega_2, C^{\mathbf{0}}(\mathbb{R}^k)_{\Omega_1 \times \Omega_2})$ is a diffeomorphism. Therefore $\Omega_1 \times \Omega_2$ is open in \mathbb{R}^k , so Ω_1 and Ω_2 are open in \mathbb{R}^m and \mathbb{R}^n , respectively.

In view of remark (4.14), nice groupoids can be considered as <u>so</u>me far-reaching generalization of foliations, and thus, the subsequent theorem (4.29) - as a generalization of Frobenius' theorem.

First, we recall the definition of a d.s. of the class \mathfrak{D}_0 . Following P.G.Walczak [22], we denote by \mathfrak{D}_0 the largest class \mathfrak{D} of d.s.'s, fulfilling the conditions:

(i) the class of d.man.'s is contained in \mathfrak{D} ,

(ii) if $(M,C)\in \mathfrak{D}$, then dim $\mathbb{T}_{\mathbf{x}}(M,C)<\infty$ for each $\mathbf{x}\in M$,

(iii) if (M,C), (M',C')eD, f:(M,C) \rightarrow (M',C') is a smooth mapping and, for some xeM, the differential $f_{*x}:T_x(M,C) \rightarrow T_{f(x)}(M',C')$ is an isomorphism, then there exists a nbh U of x open in τ_C such that fIU:(U,C_U) \rightarrow (f[U],C'_{f[U]}) is a diffeomorphism.

P.G.Walczak [23] proved the following

<u>THEOREM</u>. A d.s. (M,C) belongs to \mathfrak{D}_{0} iff, for any x \in M, there exists its nbh U $\in \mathfrak{T}_{C}$ and a d.man. \widetilde{M} such that U is contained in the support of \widetilde{M} , dim \widetilde{M} = dim $T_{x}(M,C)$ and $C_{U} = C^{\mathfrak{Q}}(\widetilde{M})_{U}$.

The class \mathfrak{D}_{o} is closed w.r.t. proper d.subs., i.e. if $(M,C)\in\mathfrak{D}_{o}$ and $A \subset M$, then $(A, C_{A})\in\mathfrak{D}_{o}$ (see [23] and [6]). Thus, the space of the smooth groupoid Φ^{R} , constructed in theorem (2.5), is of the class \mathfrak{D}_{o} . (4.29) THEOREM. A generalization of Frobenius' theorem.

Let (1.1) be any Pradines-type groupoid such that

(i) the d.s. Φ is of the class \mathfrak{D}_{0} , V is paracompact,

(11) the groupoid R_{\bullet} is of Pradines-type,

(iii) $\tilde{B}_{x}: \Phi_{x} \rightarrow (R_{\Phi})_{x}, x \in V$, are submersions,

Then (1.1) is a nice groupoid.

<u>Proof</u>. Making use of a local extension of Φ to a d.man. and of some fact from the theory of fifferential equations, one can prove (4.30) LEMMA. Let xeV, and let $\xi_1, \ldots, \xi_k \in \text{Sec A}(\Phi)$ constitute a basis of Sec A(Φ) over a nbh WGV of x, such that (a) $\tilde{B}_* \circ \xi_1, \ldots, \tilde{B}_* \circ \xi_n$, n= =dim V, constitute a basis of $\mathfrak{X}(V)$ over W, (b) $\tilde{B}_* \circ \xi_{n+1}, \ldots, \tilde{B}_* \circ \xi_k = 0$. Then there exists $\xi > 0$, K > 0 and an open nbh UC W of x, such that Exp: $X (-K,K) \times U \rightarrow \Phi$, (a,y) $\mapsto \varphi_{v,a}(\varepsilon)$,

 $\exp: \overset{n}{\times} (-K, K) \times U \longrightarrow R_{\mathfrak{F}} \subset \mathbb{V} \times \mathbb{V}, \quad (a, y) \longmapsto \psi_{v, a}(\varepsilon),$

are diffeomorphisms onto their images Ω_{II} and Ω_{OII} (with d.str.'s induced from Φ and V×V, respectively) and the diagram

$$\begin{array}{c} \overset{\mathsf{K}}{\times} (-\mathsf{K},\mathsf{K}) \times \mathsf{U} \to \Omega_{\mathsf{U}} \\ \downarrow_{\mathrm{pr}} \qquad \downarrow (\alpha, \beta) \\ \overset{\mathsf{n}}{\times} (-\mathsf{K},\mathsf{K}) \times \mathsf{U} \to \Omega_{\mathrm{oU}} \end{array}$$

commutes, where $\varphi_{y,a}(\cdot)$ and $\psi_{y,a}(\cdot)$ denote the integral curves of the vector fields k, n vector fields

$$\sum_{i=1}^{\infty} a^{\perp} \xi_{i} \quad \text{and} \quad \sum_{i=1}^{\infty} a^{\perp} \tilde{B}_{*} \xi_{i},$$

passing through $\boldsymbol{u}_{_{\boldsymbol{v}}}$ and y, respectively. \blacksquare

According to that lemma we find a covering $\mathcal{U} = \{ U_s \}_{s \in S}$ and families of d.man.'s { Ω_{U} ; U $\in \mathcal{M}$ }, { Ω_{OU} ; U $\in \mathcal{M}$ }. Let \mathcal{M} be any open covering of V su h that \overline{U}' are compact for $U' \in \mathcal{U}'$ and $\overline{\mathcal{U}}' := \{\overline{U}'; U' \in \mathcal{U}'\}$ is a refinement of a covering U.

Next, making use of strong paracompactness of V we refine starli-Let

kely \mathcal{U} to some open starlike finite covering $\mathcal{U} = \{W_t\}_{t \in T^{\circ}}$ $St(W_t; \mathcal{U}) = \bigcup_{i=1}^{m} W_{t_i} \subset U'_{s'(t)} \subset U'_{s'(t)} \subset U_{s(t)}, \quad m = m(t), \quad t \in T.$ In this connection In this connection,

 $W_t \subset U_{s(t_1)} \cap \ldots \cap U_{s(t_2)}$

and if $W_t \cap W_t \neq \emptyset$ then $W_t \cup W_t \leftarrow U_{s(t)} \cap U_{s(t')}$ For each teT we take the basis of SecA(Φ) over W_t consisting of all restrictions to W_t of all elements of the basis over $U_{s(t)}$ considered in lemma (4.30). Making several times use of the same fact from differential equations (this time to the manifolds $\Omega_{\mathrm{U}_{\mathrm{S}}}$) we obtain:

For each teT there is $K_t > 0$ such that the mappings (defined as above) Exp: $\overset{K}{\times}(-K_t, K_t) \times W_t \rightarrow \Phi$ and exp: $\overset{n}{\times}(-K_t, K_t) \times W_t \rightarrow R_{\Phi} \subset V \times V$ are diffeomorphisms onto their images Ω_t , Ω_{ot} , $(\alpha,\beta):\Omega_t \to \Omega_{ot}$ is a submersion, $\Omega_{t} \subset \Omega_{U}$ $\cap \ldots \cap \Omega_{U}$ and $\Omega_{ot} \subset \Omega_{oU_{s(t_{a})}} \cap \cdots \cap \Omega_{u_{s(t_{a})}}$. $\cap \Omega_{oU_{s(t_{a})}}$. Therefore, if $\mathbb{W}_{t} \cap \mathbb{W}_{t}$, $\neq \emptyset$ then

folds of the k-dim d.man. $\Omega_{\mathrm{U}_{\mathrm{S}(4)}}$, so they are open in the last manifold. This implies that $\Omega_t \sigma \tilde{\Omega}_t \cup \Omega_t$, and $\Omega_{ot} \sigma \Omega_{ot} \cup \Omega_{ot}$. Then $\Omega_{t} = \Omega_{t} = \bigcup_{t} \Omega_{t}$ and $\Omega_{ot} = \bigcup_{t} \Omega_{ot}$.

Hence Ω and $\Omega_{_{0}}$ satisfy the required conditions.

REFERENCES

JAN KUBARSKI

- 1. DIEUDONNE J. "Treatise on anylysis" Vol IV. Academic Press, New York and London, 1974.
- EHRESMANN CH. "Les connexions infinitesimales dans un espace fibre differentiable", Colloq. Topologie (Bruxelles 1950), Liege 1951.
- 3. EHRESMANN CH. "Categorie topologiques et categories differentiables", Colloq. Geom. Differ. Globale, Bruxelles 1959.
- 4. GREUB W, HALPERIN S, and VANSTONE R, "Connections, Curvature, and Cohomology" Vol I, Academic Press New York and London 1972
- KOWALCZYK A. "Tangent differential spaces and smooth forms", Dem. Math. 13 (1980) No 4, p.893-905.
- 6. KOWALCZYK A, and KUBARSKI J. "A local property of the subspaces of Euclidean differential spaces", Dem. Math. Vol 11, No 4, 1978, p.876-885.
- KUBARSKI J. "Smooth groupoids over foliations and their algebroids. Part I"Institute of Mathematics, Technical University of Łódź, Preprint Nr 1, May 1986.
- KUBARSKI J. "Pradines-type groupoids over foliations; cohomology, connections and the Chern-Weil homomorphism", Institute of Mathematics, Technical University of Łódź, Preprint Nr 2, August 1986.
- 9. KUBARSKI J. "Characteristic classes of some Pradines-type groupoids and a generalization of the Bott Vanishing Theorem", Proceedings of the Conference on Differential Geometry and Its Applications, August 24-30, 1986, Brno, Czechoslovakia.
- KUBARSKI J. "Some generalization of Godement's theorem on division", In this Proceedings.
- PRADINES J. "Theorie de Lie pour les groupoides differentiables. Relations entre proprietes locales et globales", C.R.A. S., Paris, t. 263, 907-910, 1966.
- PRADINES J. "Theorie de Lie pour les groupoides differentiables. Calcul differential dans la categorie des groupoides infinitesimaux", C.R.A.S., Faris, t. 264, 245-247, 1967.
- PRADINES J. "Theorie de Lie pour les groupoides differentiables", Atti Conv. Int. Geom. Dif. Bologna, 1967.
- 14. PRADINES J. "Geometrie differentielle au-dessus d'un groupoide", C.R.A.S., Faris, t. 266, 1194-1196, 1968.
- 15. PRADINES J. "Troisieme theoreme de Lie pour les groupoides differentiables", C.R.A.S., Paris, t.267, 21-23, 1968.
- 16. PRADINES J. "Au coeur de l'oeuvre de Chrles Ehresmann et de la geometrie differentielle: Les groupoides differentiables"

in Charles Echresmann "Oeuvres completes et commentees, I",

- QUE N. V. "Du prolongement des espaces fibres et des structures infinitesimales", Ann. Inst. Fourier, Grenoble, 17, 1. 1967, p. 157-223.
- SERRE J. P. "Lie algebras and Lie groups", New York-Amsterdam- Benjamin, 1965.
- SIKORSKI R. "Abstract covariant derivative", Coll. Math. 18, 1967, p. 251-272.
- SIKORSKI R. "Differential modules", Coll. Math. 24, 1971, p.45-79.
- SIKORSKI R. "Introduction to differential geometry" (in Polish) Warszawa 1972.
- 22. WALCZAK P. G. "A theorem on diffeomorphism in the category of differential spaces", Bull. Acad.Polon.Sci., Ser.Sci.Math. Astron. Phys., 21, 1973, p.325-329.
- 23. WALCZAK P. G. "On a class of differential spaces satisfying the theorem on diffeomorphism. I, II", ibidem, 22, 1974, p. 805-820.
- 24. WALISZEWSKI W. "On differential spaces the Cartesian product of which is a differentiable manifold", Coll.Math. in printing.

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