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PRADINES-TYPE GROUPOIDS

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ABSTRACT. This paper is devoted to applications of the theory of differential spaces in the sense of R.Sikorski to groupoids. By using these spaces, the notion of a smooth groupoid, much more general than a differential groupoid, is defined here. The theory of foliations is the source of such groupoids. Next, J.Pradines' idea of constructing, for every diff. groupoid, some vector bundle with natural algebraic structures - called the Lie algebroid of this diff. groupoid - is used for smooth groupoids.

INTRODUCTION. The notion of a differential groupoid introduced by Ch.Ehresmann [3] is a natural extension of the notion of a Lie group. Differential groupoids (especially Lie groupoids) constitute an appropriate direction for the development of certain geometric theories such as connexions and Lie pseudogroups. The works by J.Pradines [11] ÷ [15] were the landmark in the theory of diff. groupoids. The author defined, for each differential groupoid Φ (over a manifold V), some object - called the Lie algebroid of Φ - which is a vector bundle $T^{\alpha}_{0}\Phi$ over V such that $(T^{\alpha}_{0}\Phi)_{|x}=T_{u_x}\Phi_x$, $\Phi_x=\alpha^{-1}(x)$, $v\in V$, α-the source, u_x - the unit over x. $T^{\alpha}_{0}\Phi$ has the property: there exists some natural bijection between the module of global smooth sections of this bundle and the module of smooth right-invariant vector fields on Φ. It enables one to carry an $\mathbb{R}$-Lie algebra structure to the module of all global sections of the bundle $T^{\alpha}_{0}\Phi$. This notion generalizes the notion of a Lie algebra of a Lie group. Some new directions of the development of the theory of groupoids are described by J.Pradines in [16]. The theory of foliations (also that of pseudogroups) is the source of the important nontransitive groupoids whose space is not - in general - a manifold. For example, the subgroupoid $\Phi^{\mathcal{F}}$ [9] of a Lie groupoid Φ (over V) consisting of the elements for which the source and the target lie on some leaf of a given foliation $\mathcal{F}$ of V. However, it is evident that one can always define on $\Phi^{\mathcal{F}}$ some natural structu-

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re of a differential space in the sense of R.Sikorski [19] (see also [20], [21]), and it turns out that all operations in $\Phi^{\mathfrak{F}}$ are then smooth as the mappings in the category of differential spaces. This gives rise to the defining of the notion of a groupoid in the category of diff. spaces. If, in addition, the sets $\alpha^{-1}(x)$, $x\in V$ (α - the source in a given groupoid Φ over V) are the so-called leaves of the diff. space Φ, then this groupoid is called a smooth groupoid. $\Phi^{\mathfrak{F}}$ is such a groupoid.

The present author's observation show that, by following the idea of J.Pradines, one can construct, for each smooth groupoid Φ, an object $A(\Phi)$, analogous to the Lie algebroid of a diff. groupoid, not being - unfortunately in general - a vector bundle. The above example $\Phi^{\mathfrak{F}}$ of a smooth groupoid have the property that the constructed object $A(\Phi^{\mathfrak{F}})$ is a vector bundle (although $\Phi^{\mathfrak{F}}$ is hardly ever diff. groupoid). The smooth groupoids Φ for which the objects $A(\Phi)$ are vector bundles shall be call the Pradines-type groupoids. An especially important role will be played by those groupoids from among them which are also the so-called smooth groupoids over foliations. They are - in the author's opinion - a proper generalization of principal fibre bundles, for they enable one to build a sensible theory of connexions (see [8], [9]).

This work (in the considerable part) has come into being on the basis of preprint [7] and is its extension.

1. PRELIMINARIES. First, we give two definitions fundamental for our work: of a groupoid and of a differential space.

By a groupoid we shall mean (after N.V.Que [17]) the system

(1.1) $$(\Phi,\alpha,\beta,V,\cdot)$$

consisting of sets Φ and V and mappings $\alpha,\beta:\Phi\longrightarrow V$, $\cdot:\Phi*\Phi\longrightarrow\Phi$ where $\Phi*\Phi=\{(g,h)\in\Phi\times\Phi;\ \alpha g=\beta h\}$, fulfilling the axioms (i) $\alpha(g\cdot h)=\alpha h$ and $\beta(g\cdot h)=\beta g$ for $(g,h)\in\Phi*\Phi$, (ii) $(f\cdot g)\cdot h=f\cdot(g\cdot h)$ for (f,g), (g,h) $\in\Phi*\Phi$, (iii) for each point $x\in V$, there exists an element $u_x\in\Phi$ such that $\alpha(u_x)=\beta(u_x)=x$, $h\cdot u_x=h$ when $\alpha h=x$, $u_x\cdot g=g$ when $\beta g=x$ (u_x is uniquely determined and called the unit over x), (iv) for each element $h\in\Phi$, there exists an element $h^{-1}\in\Phi$ such that $\alpha(h^{-1})=\beta h$, $\beta(h^{-1})=\alpha h$, $h\cdot h^{-1}=u_{\beta h}$, $h^{-1}\cdot h=u_{\alpha h}$ (h^{-1} is uniquely determined).

By a differential space (d.s. for short) (see R.Sikorski [19]÷[21]) we mean each couple (M,C) (sometimes denoted briefly by M) consisting of a set M and a non-empty family C of real functions on M closed with respect to (w.r.t.) localization and superposition with all functions of C^{∞}-class on the Catesian spaces. The set C is then called the differential structure of this space (d.str. for short) and M - its support. More precisely, let us denote by τ_C the weakest topology on M such that all functions of C are continuous. For $A\subset M$, we denote by C_A the set of all functions $h:A\to\mathbb{R}$ such that, for any $x\in A$, there exist a neighbourhood (nbh) $U\in\tau_C$ of x and a function $g\in C$ satisfying $h|U\cap A=g|U\cap A$. The closedness w.r.t. localization may be expressed in the form $C_M=C$. Denote by $\operatorname{sc} C$ the set of all functions $\varphi(g_1(\cdot),\dots,g_m(\cdot))$ where φ is a C^∞-function on $\mathbb{R}^m$, $g_i\in C$, $i\leq m$, $m=1,2,\dots$ The closedness w.r.t. superposition with all functions of C^∞-class on the Cartesian spaces means that $\operatorname{sc} C=C$.

Every d.s. (M,C) is also considered as the topological space (M,τ_C). If $\hat{C}$ is a non-empty family of real functions on M, then $C:=(\operatorname{sc}\hat{C})_M$ is the smallest d.str. on M containing $\hat{C}$. C is called the d.str. generated by $\hat{C}$. If (M,C) is a d.s., then (A,C_A) is such a space, too, for any subset $A\subset M$ and is called a proper differential subspace of (M,C) (proper d.subs. for short). C_A is called induced from (M,C) on A. (A,C_A) is sometimes denoted by $M_{|A}$.

Let (M,C) and (N,D) be any d.s.'s. The mapping $f:M\to N$ is called (i) smooth if $g\circ f\in C$ for $g\in D$. Then we write $f:(M,C)\to(N,D)$, (ii) a diffeomorphism (diff. for short) if it is a bijection and f and f^{-1} are smooth, (iii) an embedding if $f:(M,C)\to(f[M],D_{f[M]})$ is a diff. By a product $(M,C)\times(N,D)$ we mean the d.s. $(M\times N,C\times D)$ where $C\times D$ is the d.str. generated by $\{g\circ pr_1,\ g\in C\}\cup\{h\circ pr_2,\ h\in D\}$. For $A\subset M$ and $B\subset N$ we have $(C\times D)_{A\times B}=C_A\times D_B$.

Let V be any C^∞-manifold. Then $(V,C^\infty(V))$ is, of course, a d.s. and $\tau_{C^\infty(V)}\subset \operatorname{Top} V$. What is more, the equality $\tau_{C^\infty(V)}=\operatorname{Top} V$ holds if and only if (iff) V is Hausdorff. In the Hausdorff case, for any open set $U\subset V$ (the notation: $U\Subset V$), we have $C^\infty(V_{|U})=C^\infty(V)_U$. In this connection, we adopt the following definition [19]: a d.s. (V,C) is called an n-dim. differential manifold (d.man.) if, for each point $x\in V$, there exist a nbh $U\in\tau_C$ of x, an open subset $\Omega\subset\mathbb{R}^n$ and some diff. $\varphi:(U,C_U)\to(\Omega,C^\infty(\mathbb{R}^n)_\Omega)$. The topology τ_C is then Hausdorff. We shall identify a Hausdorff C^∞-manifold V with the d.man. $(V,C^\infty(V))$.

Having d.s.'s at our disposal, we are able to give the following

(1.2) DEFINITION. By a groupoid in the category of d.s.'s we mean groupoid (1.1) in which Φ and V are d.s.'s and the mappings $\alpha,\beta:\Phi\to V$, $^{-1}:\Phi\to\Phi$, $h\mapsto h^{-1}$, $u:V\to\Phi$, $x\mapsto u_x$, as well as $\cdot:\Phi*\Phi\to\Phi$ are smooth ($\Phi*\Phi$ denotes here the proper d.subs. of $\Phi\times\Phi$).

We notice that $u:V\to\Phi$ is an embedding.

From now, by a groupoid we shall mean a groupoid in the category of d.s.'s and we shall sometimes say "a groupoid Φ" instead of "a groupoid $(\Phi,\alpha,\beta,V,\cdot)$".

Homomorphisms between groupoids are defined in an evident manner.

(1.3) EXAMPLE. Let $R\subset V\times V$ be any equivalence relation on a d.s. V. Then the system

$$(R,pr_1|R,pr_2|R,V,\cdot), \tag{1.4}$$

where R is here the proper d.subs. of $V\times V$ and $(y,z)\cdot(x,y)=(x,z)$, is a groupoid called the groupoid of the equivalence relation R. ■

(1.5) EXAMPLE. Let Γ be any pseudogroup of smooth transformations on a d.man. V. Then, for each $k=1,2,\dots$, the set of jets $\{j^k_x f;\ f\in\Gamma, x\in D_f\}$ $\subset J^k(V,V)$, with the d.str. induced from $J^k(V,V)$, forms a groupoid. ■

Groupoid (1.1) is said to be a differential groupoid [3] if Φ and V are d.man.'s and $\alpha,\beta:\Phi\to V$ are submersions. A diff. groupoid is said to be a Lie groupoid [17] if it is transitive. A principal fibre bundle P determines the Lie groupoid of Ehresmann PP^{-1} [2].

2. SMOOTH GROUPOIDS. The notion of a subspace of a d.s. can be found in [19] but it is too strong for us, so we adopt the following:

A d.s. (N',D') is said to be a differential subspace (d.subs. for short) of a d.s. (N,D) if $N'\subset N$, and for each point $y\in N'$, there exists a nbh $U\in\tau_{D'}$ of y such that $D'_U=D_U$. Then we write $(N',D')\hookrightarrow(N,D)$. If $D'=D_{N'}$ then (N',D') is a proper d.subs. of (N,D).

Any immerse d.subman. of a d.man. is an example of a d.subs.

By a leaf (k-leaf) of a d.s. (M,C) we mean a subset $L\subset M$ if there exists a d.str. D on L such that (L,D) is a d.man. (of dimension k), (L,D) is a d.subs. of (M,C) and, for each locally arcwise connected topological space X and a continuous mapping $f:X\to(M,\tau_C)$ such that $f[X]\subset L$, the induced mapping $\hat{f}:X\to(L,\tau_D)$, $x\mapsto f(x)$, is continuous, too. We notice that

(1) the d.str. D on L is uniquely determined,

(2) each connected component of (L,D) is equal to an arcwise connected component of the subset L in (M,τ_C),

(3) L is a leaf iff any of its arcwise connected components is a leaf,

(4) if (X,E) is any d.s. whose topology τ_E is locally arcwise connected, then, for each smooth mapping $f:(X,E)\to(M,C)$ such that $f[X]\subset L$, the mapping $\hat{f}:(X,E)\to(L,D)$, $x\mapsto f(x)$, is also smooth.

Sometimes, the manifold (L,D) will be called a leaf of (M,C).

Each element of a foliation of a d.man. is an example of a leaf.

Now, we give the fundamental

(2.1) DEFINITION. By a smooth groupoid we mean groupoid (1.1) in which the sets $\alpha^{-1}(x)$, $x\in V$, are leaves of the d.s. Φ.

The set $\alpha^{-1}(x)$ equipped with the suitable d.man. structure is called the <u>leaf of this groupoid over x</u> and denoted by Φ_x, $x \in V$.

The mappings

$$D_h : \Phi_{\beta h} \to \Phi_{\alpha h},\ g \mapsto g \cdot h, \tag{2.2}$$

$h \in \Phi$, are diff.'s.

Every diff. groupoid is, of course, a smooth groupoid; the proper d.subman. $\alpha^{-1}(x)$ of Φ is a leaf over x, $x \in V$.

<u>(2.3) PROPOSITION</u>. Groupoid (1.4) is a smooth groupoid iff each abstract class of R is a leaf of V. In this case, the mapping

$$\gamma_x : L_x \to R_x,\ y \mapsto (x,y), \tag{2.4}$$

is a diff. of leaves (L_x - the abstract class of R through x). ■

<u>(2.5) THEOREM</u>. Let (1.1) be any Lie groupoid. Then, for an equivalence relation R for which (1.4) is a smooth groupoid, the subgroupoid

$$\Phi^R = (\alpha,\beta)^{-1}[R] \tag{2.6}$$

equipped with the d.str. of a proper d.subs. of Φ, turns out to be a smooth groupoid.

<u>Proof</u>. Of course, Φ^R forms a groupoid. Let $x \in V$. Consider the submersion $\beta_x : \Phi_x \to V$, $h \mapsto \beta h$, and take the abstract class L_x of R through x. The inverse image $\beta_x^{-1}[L_x]$ forms in a natural manner an immerse d.subman. Φ_x^R of the d.man. Φ_x, characterized by the property:

- if $A \subset L_x$ and the d.man. $L_{x|A}$ is a proper d.subman. of V, then $\beta_x^{-1}[A] \subset \Phi_x^R$ and the d.man. $\Phi^R_{x|\beta_x^{-1}[A]}$ is a proper d.subman. of Φ_x. Of course, $\beta_x^R : \Phi_x^R \to L_x$ - the induced mapping - is a submersion. We have to show that Φ_x^R is a leaf of the d.s. Φ^R, which is equivalent to the fact that Φ_x^R is a leaf of the d.man. Φ_x.

Let X be any locally arcwise connected topological space and $f : X \to \Phi_x$ - any continuous mapping such that $f[X] \subset \Phi_x^R$. Take an arbitrary point $t_0 \in X$. By the submersivity of β_x, there exist nbh's $W \subset \Phi_x$ and $\tilde{W} \subset V$ of $f(t_0)$ and $y_0 := \beta(f(t_0))$, respectively, and a diff. $\psi : W \to \tilde{W} \times \mathbb{R}^s$, $s = \operatorname{codim} L_x$, such that $\psi^1 : W \to \tilde{W}$, $h \mapsto pr_1 \circ \psi(h)$, is equal to $\beta_x | W$. Take any subset $\tilde{U} \subset L_x$, containing y_0, such that $L_{x|\tilde{U}}$ is a proper d. subman. of V. Put $U = \psi^{-1}[\tilde{U} \times \mathbb{R}^s]$. Of course, $U = (W \cap \Phi_x^R) \cap \beta_x^{-1}[\tilde{U}] \subset \Phi_x^R$ and $\varphi : \Phi^R_{x|U} \to L_{x|\tilde{U}} \times \mathbb{R}^s$, $h \mapsto \psi(h)$, is a diff. Let $B = f^{-1}[W]$. Then $f[B] \subset W \cap \Phi_x^R$, so the image of the mapping $\psi^1 \circ f|B$ (which is equal to $\beta_x \circ f|B$) is contained in $\tilde{W} \cap L_x$. From the assumption about R it follows that $\psi^1 \circ f|B : B \to L_x$ is continuous. Put $\tilde{B} = (\psi^1 \circ f|B)^{-1}[\tilde{U}]$. $\tilde{B} \in \operatorname{Top} X$ and $\psi^1 \circ f|\tilde{B} : \tilde{B} \to L_{x|\tilde{U}}$ is continuous, so is $f|\tilde{B} = \varphi^{-1} \circ (\psi^1 \circ f|\tilde{B}, \psi^2 \circ f|\tilde{B}) : \tilde{B} \to \Phi^R_{x|U}$. The free choice of $t_0 \in X$ implies that $f : X \to \Phi_x^R$ is continuous. ■

<u>3. ALGEBROID OF A SMOOTH GROUPOID</u>. The present author's observa-

tions show that by following the idea of J.Pradines [12],[13], one can construct an object analogous to the Lie algebroid of a diff. groupoid for groupoids from a much wider class, namely for smooth groupoids.

We first recall the notion of a tangent vector and the tangent d. s. over a d.s. By a <u>tangent vector to a d.s. (M,C) at $x\in M$</u> [19] we mean each linear mapping $v:C\to\mathbb{R}$ such that $v(f\cdot g)=v(f)\cdot g(x)+f(x)\cdot v(g)$ for $f,g\in C$. All tangent vectors at x form a vector space which is denoted by $T_x(M,C)$ and called the <u>tangent vector space at x</u>. By the <u>differential at x</u> [20] of any smooth mapping $f:(M,C)\to(N,D)$ between d.s.'s (M,C) and (N,D) we mean the linear mapping $f_{*x}:T_x(M,C)\to T_{f(x)}(N,D)$ defined by the formula $f_{*x}(v)(g)=v(g\circ f)$, $g\in D$, $v\in T_x(M,C)$. If (N',D') is a d.subs. of (N,D) and $i:(N',D')\hookrightarrow(N,D)$ denotes the inclusion, then, for $y\in N'$, $i_{*y}:T_y(N',D')\to T_y(N,D)$ is a monomorphism; with its help the space $T_y(N',D')$ is identified with the vector subspace $\operatorname{Im} i_{*y}$ of $T_y(N,D)$.

Let (M,C) be any d.s. We put

(i) $T(M,C)=\bigsqcup_{x\in M}T_x(M,C)$ (the disjoint union of all tang. spaces),

(ii) $\pi:T(M,C)\to M$ - the canonical projection,

(iii) $TC=(\mathrm{sc}\,\hat{C})_{T(M,C)}$ where $\hat{C}=\{g\circ\pi;\ g\in C\}\cup\{dg;\ g\in C\}$ ($dg:T(M,C)\to\mathbb{R}$, $v\mapsto v(g)$).

Following A.Kowalczyk [5], the d.s. $(T(M,C),TC)$ is called the <u>tangent d.s. to a d.s. (M,C)</u>. Any (smooth) section $X:M\to T(M,C)$ of π is called a (<u>smooth</u>) <u>vector field on (M,C)</u>. The smoothness of X is equivalent to the fact that $X(f)\in C$ for $f\in C$. We shall denote the C-module of all smooth vector fields on (M,C) by $\mathfrak{X}(M,C)$.

In the sequel of this chapter, we fix a smooth groupoid (1.1). Let C and D denote the d.str.'s of Φ and V, respectively. This determines the system

$$(A(\Phi),p,V) \tag{3.1}$$

in which (1) $A(\Phi)$ is the proper d.subs. of the tangent d.s. $T\Phi$ with the support $\bigsqcup_x T_{u_x}\Phi_x\subset T\Phi$ (we recall that Φ_x denotes the leaf of Φ over x, $x\in V$), (2) $p:A(\Phi)\to V$ is the projection defined by $p(v)=x$ iff $v\in T_{u_x}\Phi_x$. Of course, p is smooth. The structure of a vector space is defined in each fibre of p. Unfortunately, system (3.1) is not - in general - a vector bundle (even if V is a manifold). However, it has some interesting properties described in theorem (3.7) below.

A vector field X on Φ is called an <u>α-field</u> if $X_h\in T_h\Phi_{\alpha h}$, $h\in\Phi$. An α-field X is called <u>right-invariant</u> (briefly r-i) if $(D_h)_{*g}(X_g)=X_{g\cdot h}$, $g,h\in\Phi$ and $\alpha g=\beta h$, where D_h are mappings (2.2). It is easy to see that the Lie bracket of smooth r-i vector fields is such a field, too. All smooth r-i vector fields on Φ form an $\mathbb{R}$-Lie algebra (w.r.t. the Lie bracket) and a D-module (w.r.t. the multiplication $f\cdot X:=f\circ\beta\cdot X$) denoted by $\mathfrak{X}^R(\Phi)$. $\mathrm{Sec}A(\Phi)$ - the vector space of all global smooth sections of p - forms a D-module, too.

Each smooth r-i vector field X determines a smooth section

$$X_o: V \longrightarrow A(\Phi),\ x \longmapsto X_{u_x},$$

of p. The mapping

$$(3.2)\qquad \mathfrak{X}^R(\Phi) \longrightarrow \mathrm{Sec}\,A(\Phi),\ X \longmapsto X_o,$$

is a homomorphism of D-modules. Conversely, we have the following

<u>(3.3) PROPOSITION</u>. For each $\eta\in\mathrm{Sec}\,A(\Phi)$, there exists exactly one smooth r-i vector field on Φ, denoted by η', such that

$$(*)\qquad \eta'_{u_x}=\eta_x,\ x\in V.$$

The mapping

$$(3.4)\qquad \mathrm{Sec}\,A(\Phi)\longrightarrow\mathfrak{X}^R(\Phi),\ \eta\longmapsto\eta',$$

is an isomorphism of D-modules, inverse to (3.2).

<u>Proof</u>. Let $\eta\in\mathrm{Sec}\,A(\Phi)$. A r-i vector field η' on Φ, such that $(*)$ holds, is defined by the formula

$$(**)\qquad \eta'_h=(D_h)_{*u_{\beta h}}(\eta_{\beta h}),\ h\in\Phi,$$

which proves the uniqueness. To show the existence, we must prove that the vector field η' defined by $(**)$ is r-i and smooth. This first fact is easy to see. To prove the second, we take an arbitrary $f\in C$. For $h\in\Phi$, we have $\eta'_h(f)=\eta\circ\beta(h)(\Phi_{\beta h}\ni g\longmapsto f\circ(\cdot)(g,h))$. From the assumption that $\cdot:\Phi*\Phi\longrightarrow\Phi$ is smooth we have $f\circ(\cdot)\in(C\times C)_{\Phi*\Phi}$. We fix $h_o\in\Phi$ and find a nbh $\Omega\in\tau_{C\times C}$ of $(u_{\beta h_o},h_o)$ and a function $\tilde f\in C\times C$ such that $f\circ(\cdot)|\Omega\cap(\Phi*\Phi)=\tilde f|\Omega\cap(\Phi*\Phi)$. Thus, for h from some nbh of h_o, we have $\eta'_h(f)=\eta\circ\beta(h)(\tilde f(\cdot,h))$. The function $\Phi\ni h\longmapsto Y(h)(\tilde f(\cdot,h))$, where $Y:\Phi\longrightarrow T\Phi$, $g\longmapsto\eta\circ\beta(g)$, belongs to C; see the lemma below. ∎

<u>(3.5) LEMMA</u>. Let $(M,\tilde C)$ and $(N,\tilde D)$ be any d.s.'s and $Y:(N,\tilde D)\longrightarrow(T(M,\tilde C),T\tilde C)$ - any smooth mapping. Then, for any smooth function $\tilde f\in\tilde C\times\tilde D$, the following function $N\ni x\longmapsto Y(x)(\tilde f(\cdot,x))\in\mathbb{R}$ belongs to $\tilde D$.

<u>Proof</u>. Put $Y_1:N\longrightarrow T(M\times N)$, $x\longmapsto(Y(x),\theta_x)$, where $\theta_x\in T_xN$ denotes the null vector. We prove that $Y_1:(N,\tilde D)\longrightarrow(T(M\times N),T(\tilde C\times\tilde D))$ is smooth. $\tilde C\times\tilde D$ is generated by $E=\{g\circ pr_1;\ g\in\tilde C\}\cup\{h\circ pr_2;\ h\in\tilde D\}$, so [5] $T(\tilde C\times\tilde D)$ is generated by $\tilde E=\{\delta\circ\pi;\ \delta\in E\}\cup\{d\delta;\ \delta\in E\}$. To see the smoothness of Y_1, we have to notice only that $\zeta\circ Y_1\in\tilde D$ for $\zeta\in\tilde E$. In the end, we see that $(x\longmapsto Y(x)(\tilde f(\cdot,x)))=d\tilde f\circ Y_1\in\tilde D$. ∎

<u>(3.6) DEFINITION</u>. By a <u>vector pseudobundle</u> (<u>over a d.s. V</u>) we mean each system (A,p,V) containing d.s.'s A and V and a surjective smooth mapping $p:A\longrightarrow V$ in whose fibres some structures of vector spaces are defined and the following properties hold:

(1) $+:A\oplus A\longrightarrow A$, $(v,w)\longmapsto v+w$, $\cdot:\mathbb{R}\times A\longrightarrow A$, $(r,v)\longmapsto r\cdot v$, are smo-

oth mappings where $A\oplus A$ denotes the proper d.subs. of $A\times A$ with the support $\{(v,w)\in A\times A;\ pv=pw\}$,

(2) for any number $m\in N$, any smooth sections $\xi_1,\dots,\xi_m$ of p and any set $U\subset V$ (not necessarily open) such that the vectors $\xi_1(x),\dots,\xi_m(x)$ are linearly independent for $x\in U$, the mapping

$$\varphi: V_{|U}\times\mathbb{R}^m \longrightarrow A,\ (x,a)\longmapsto \textstyle\sum_i a^i\xi_i(x),$$

is a diff. onto its image.

Let (A,p,V) and (A',p',V') be two vector pseudobundles. By a <u>homomorphism</u> between them we mean a pair of smooth mappings (h,H), $h:V\to V'$, $H:A\longrightarrow A'$, such that $p'\circ H=h\circ p$ and for each $x\in V$, $H_{|x}:A_{|x}\longrightarrow A'_{|h(x)}$ is a linear homomorphism of vector spaces. If $V=V'$ and $h=\mathrm{id}$, then this homomorphism is called <u>strong</u> and denoted by one letter H. All vector pseudobundles and homomorphisms form a category.

<u>(3.7) THEOREM</u>. System (3.1) is a vector pseudobundle.

<u>Proof</u>. Extending sections $\xi_1,\dots,\xi_m\in\mathrm{Sec}\,A(\Phi)$ to smooth vector fields on Φ (for example, to r-i vector fields), we see that the theorem is an immediate consequence of the following lemma. ■

<u>(3.8) LEMMA</u>. The tangent d.s. $T(M,\tilde{C})$ to a d.s. $(M,\tilde{C})$ is a vector pseudobundle.

<u>Proof</u>. The smoothness of + and $\cdot$ is easy to see. Now, take $m\in N$, any smooth vector fields $X_1,\dots,X_m\in\mathfrak{X}(M,\tilde{C})$ and any subset $U\subset M$ (not necessarily open) such that the vectors $X_1(x),\dots,X_m(x)$ are linearly independent for each $x\in U$ and define $\varphi: M_{|U}\times\mathbb{R}^m\longrightarrow T(M,\tilde{C})$, $(x,a)\longmapsto \sum_i a^i X_i(x)$. The smoothness of φ is evident. To prove this for φ^{-1}, we put $pr_1: U\times\mathbb{R}^m\longrightarrow U$, $(x,a)\longmapsto x$, and $p^s: U\times\mathbb{R}^m\longrightarrow\mathbb{R}$, $(x,a^1,\dots,a^m)\longmapsto a^s$, $s\leqslant m$. Of course $pr_1\circ\varphi^{-1}=\pi|\varphi[U\times\mathbb{R}^m]$ is smooth. So, it suffices to show that $p^s\circ\varphi^{-1}\in(T\tilde{C})_{\varphi[U\times\mathbb{R}^m]}$. For this purpose, we notice that $p^s\circ\varphi^{-1}(\sum_i a^iX_i(x))=a^s$, and we take any point $x_o\in U$ and functions $f^1,\dots,f^m\in C$ such that $X_i(f^j)(x_o)=\delta_i^j$ [21]. Then, for some nbh $\tilde{U}$ of x_o, we have: $\det[X_i(f^j)(x)]\neq 0$, $x\in\tilde{U}$, and we can define the mapping $\psi:\tilde{U}\longrightarrow GL(m,\mathbb{R})$, $x\longmapsto[X_i(f^j)(x)]$. Let ${}^{-1}\circ\psi(x)=[c_j^k(x);\ j,k\leqslant m]$, $x\in\tilde{U}$. For $F:\mathbb{R}^{2m}\longrightarrow\mathbb{R}$, $(x^1,\dots,x^m,y^1,\dots,y^m)\longmapsto\sum_j y_jx_j$, we have

(1) $F(c_1^s\circ\pi,\dots,c_m^s\circ\pi,df^1,\dots,df^m)\in(T\tilde{C})_{\pi^{-1}[\tilde{U}]}$,

(2) $F(c_1^s\circ\pi,\dots,c_m^s\circ\pi,df^1,\dots,df^m)|\pi^{-1}[\tilde{U}]\cap\varphi[U\times\mathbb{R}^m]$ $=p^s\circ\varphi^{-1}|\pi^{-1}[\tilde{U}]\cap\varphi[U\times\mathbb{R}^m]$ which ends the proof. ■

For $\xi,\eta\in\mathrm{Sec}\,A(\Phi)$, we put

$$[\![\xi,\eta]\!] := [\xi',\eta']_o, \tag{3.9}$$

and take the mapping $\tilde{\beta}_*: A(\Phi)\longrightarrow TV$, $v\longmapsto\beta_*v$. For any $\xi\in\mathrm{Sec}\,A(\Phi)$, the r-i vector field ξ' on Φ is β-related to the vector field $\tilde{\beta}_*\circ\xi\in\mathfrak{X}(V)$

<u>(3.10) DEFINITION</u>. The system

$$(A(\Phi),\ [\![\cdot,\cdot]\!],\ \tilde{\beta}_*) \tag{3.11}$$

is called the <u>algebroid of the smooth groupoid Φ</u>.

The fundamental properties of system (3.11) are described by

<u>(3.12) THEOREM</u>. (a) The system $(\mathrm{Sec}\,A(\Phi), [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

(b) the mapping $\mathrm{Sec}\,\tilde{\beta}_*:\mathrm{Sec}\,A(\Phi) \to \mathfrak{X}(V)$, $\xi \mapsto \tilde{\beta}_*\circ\xi$, is a homomorphism of Lie algebras,

(c) the equality $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\tilde{\beta}_*\circ\xi)(f)\cdot\eta$ holds for $\xi,\eta \in \mathrm{Sec}\,A(\Phi)$ and $f\in D$.

<u>Proof</u>. (a) is easy to see, (b) Let $\xi,\eta\in\mathrm{Sec}\,A(\Phi)$. Since ξ' is β-related to $\tilde{\beta}_*\circ\xi$ and η' - to $\tilde{\beta}_*\circ\eta$, therefore $[\xi',\eta']$ is β-related to $[\tilde{\beta}_*\circ\xi, \tilde{\beta}_*\circ\eta]$. Also, $[\![\xi,\eta]\!]'$ is β-related to $\tilde{\beta}_*\circ[\![\xi,\eta]\!]$, thus by (3.9) and by the surjectivity of β, we see that $\tilde{\beta}_*\circ[\![\xi,\eta]\!] = [\tilde{\beta}_*\circ\xi, \tilde{\beta}_*\circ\eta]$. (c) follows from the equality $[X, f\circ\beta\cdot Y] = f\circ\beta\cdot[X,Y] + (\tilde{\beta}_*\circ X_o)(f)\circ\beta\cdot Y$ for $X,Y\in\mathfrak{X}^R(\Phi)$ and $f\in D$. ∎.

The properties of system (3.11) described in theorems (3.7) and (3.12) suggest the following definition of an abstract algebroid.

<u>(3.13) DEFINITION</u>. By an <u>algebroid</u> we mean a system

$$(A, [\![\cdot,\cdot]\!], \gamma), \tag{3.14}$$

consisting of (1) a vector pseudobundle $A = (A,p,V)$, (2) a mapping $[\![\cdot,\cdot]\!]:\mathrm{Sec}\,A\times\mathrm{Sec}\,A \to \mathrm{Sec}\,A$ where $\mathrm{Sec}\,A$ denotes the D-module of all smooth sections of p (D - the d.str. of V), (3) a mapping $\gamma: A\to TV$, such that (a) the system $(\mathrm{Sec}\,A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, (b) the mapping $\mathrm{Sec}\,\gamma:\mathrm{Sec}\,A \to \mathfrak{X}(V)$, $\xi\mapsto\gamma\circ\xi$, is a homomorphism of Lie algebras, (c) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$ for $\xi,\eta\in\mathrm{Sec}\,A$, $f\in D$.

If A is a vector bundle (over a manifold V), then (3.14) is simply a <u>Lie algebroid</u> in the sense of J.Pradines [12], [13].

For two algebroids $(A, [\![\cdot,\cdot]\!], \gamma)$ and $(A', [\![\cdot,\cdot]\!]', \gamma')$ over the same d.s. V, by a (<u>strong</u>) <u>homomorphism</u> between them we mean a homomorphism of vector pseudobundles $H: A \to A'$, such that (1) $\gamma'\circ H = \gamma$, (2) $\mathrm{Sec}\,H: \mathrm{Sec}\,A \to \mathrm{Sec}\,A'$ is a homomorphism of $\mathbb{R}$-Lie algebras.

Any strong homomorphisn $F:\Phi\to\Phi'$ between two smooth groupoids determines a homomorphism of their algebroids $\tilde{F}_*: A(\Phi)\to A(\Phi')$, $v\mapsto F_*v$. The covariant functor $\Phi\mapsto A(\Phi)$, $F\mapsto\tilde{F}_*$, obtained above is called (like in J.Pradines [13] for diff. groupoids) the <u>Lie functor</u>.

It is possible that definitions (3.6) and (3.13) are too general. The answer to the following question will solve this problem.

<u>(3.15) THE FIRST OPEN PROBLEM</u>. Is each algebroid isomorphic to the algebroid of some smooth groupoid? If the answer is negative, find any necessary and sufficient conditions for the algebroid to be isomorphic.

<u>4. PRADINES-TYPE GROUPOIDS</u>.

(4.1) DEFINITION. By a groupoid of Pradines type we shall mean any smooth groupoid (1.1) in which V is a d.man. and system (3.1) is a vector bundle.

(4.2) EXAMPLES. (1) Differential groupoid (1.1) is of Pradines type. Indeed, $A(\Phi)\cong u^{*}T^{\alpha}\Phi$ where $T^{\alpha}\Phi = \operatorname{Ker}\alpha_{*}$.

(2) The groupoid of the equivalence relation R determined by a foliation $\mathcal{F}$ of a d.man. V is of Pradines type. Indeed, $A(R)\cong T\mathcal{F}$.

(3) The smooth groupoid Φ^{R}, defined in theorem (2.5) with the help of an equivalence relation R for which the family of abstract classes is a foliation $\mathcal{F}$ of V, is a groupoid of Pradines type. In fact, $A(\Phi^{R}) = \tilde{\beta}_{*}^{-1}[T\mathcal{F}]$ is a vector subbundle of $A(\Phi)$. ∎

By the remark following definition (3.13), we obtain

(4.3) COROLLARY. If (1.1) is a groupoid of Pradines type, then system (3.11) is a Lie algebroid.

(4.4) THEOREM. Smooth groupoid (1.1) in which V is a d.man. is a groupoid of Pradines type iff

(i) for each vector $v\in A(\Phi)$, there exists $\xi\in \operatorname{Sec} A(\Phi)$ such that $\xi(pv)=v$,

(ii) the function $V\ni x \mapsto \dim A(\Phi)_{|x}$ is constant.

Proof. "$\Rightarrow$" is evident, "$\Leftarrow$". Let $x\in V$ and let $(v_1,\dots,v_m)$ be any basis of $A(\Phi)_{|x}$. Take $\xi_1,\dots,\xi_m\in \operatorname{Sec} A(\Phi)$ and a nbh U of x in V such that $\xi_i(x)=v_i$, $i\leq m$, and the vectors $\xi_1(y),\dots,\xi_m(y)$ are linearly independent for $y\in U$. We need notice that

$$\varphi : U\times\mathbb{R}^{m} \longrightarrow p^{-1}[U],\ (y,a^1,\dots,a^m) \mapsto \textstyle\sum_i a^i \xi_i(y)$$

is a diffeomorphism. This is an consequence of theorem (3.7). ∎

Now, let R be any equivalence relation on a d.man. V. In the theorem below, we give the complete answer to the question when the groupoid of R is of Pradines type. We see that - because of statement (2) - this theorem may be considered as the next generalization of Godement's theorem on division (see [10]).

(4.5) THEOREM. The following conditions are equivalent:

(1) (1.4) is a groupoid of Pradines type with $\dim R_x = k$, $x\in V$.

(2) The family $\mathcal{F}$ of all arcwise connected components of all abstract classes of R is a k-dim. foliation.

(3) (a) (1.4) is a smooth groupoid, and

(b) there exists a subset $\Omega\subset R$ such that

(i) $\Delta\subset\Omega$ where $\Delta = \{(x,x)\in V\times V;\ x\in V\}$,

(ii) Ω is a proper n+k-dim. d.subman. of $V\times V$,

(iii) $pr_1|\Omega : \Omega \to V$ is a submersion,

(iv) for each $x\in V$, the proper d.subman.

$$\Omega_x := (pr_1|\Omega)^{-1}(x)$$

of Ω is an open d.subman. of the leaf R_x.

(4) There exists a subset $\Omega \subset R$ such that

(i)$\div$(iii) as above,

(iv') $D_{(x,y)}[\Omega_y] \cap \Omega_x \subset \Omega_x$ for $(x,y) \in R$,

(v') connected components of the manifold $\tilde{R}_x$, $x \in V$, (see lemma in [10]) are equal to arcwise connected components of the set R_x in $V \times V$.

Proof. (1)$\Rightarrow$(2). Let us take any abstract class L of R and $x \in L$. We define on L a d.str. of a d.man. in such a way that the mapping γ_x: $L \to R_x$, $y \mapsto (x,y)$, is a diffeomorphism. L is an immerse subman. of V. From the fact that R_x is a leaf of $V \times V$ we see that connected components of the manifold L are equal to arcwise connected components of the set L in V, so the connected component B_x of x of the manifold L is an element of $\mathcal{F}$, and $T_xB_x = (pr_2)_{*(x,x)}[A(R)_{|x}] = (pr_2|R)^{\sim}_{*x}[A(R)_{|x}]$. Since $(pr_2|R)^{\sim}_{*}: A(R) \to TV$ is a monomorphism of the bundles, we see that $T\mathcal{F} = \mathrm{Im}(pr_2|R)^{\sim}_{*}$ is a vector subbundle of TV. Thus (2) results now from some version of Frobenius' theorem [1,p.86].

(2)$\Rightarrow$(3). Condition (a) follows from the observation that each abstract class of R is a leaf of V (w.r.t. the definition contained in chapter 2). To prove (b), we take any nice covering $\{(U_i,\varphi_i,\mathbb{R}^n);\ i \in N\}$ of $\mathcal{F}$, $n = \dim V$. Let us denote by Q^i_x the plaque of the chart (U_i,φ_i) which contains x, $x \in U_i$. Like in the proof of theorem 2 from paper [10], we see that $\Omega = \bigcup_i \Omega_i$, where $\Omega_i = \{(x,y) \in V \times V;\ x \in U_i,\ y \in Q^i_x\}$, has properties (i)$\div$(iii). To show (iv), we notice that the inclusion $\Omega_x \hookrightarrow R_x$ is an immersion, and that $\dim \Omega_x = \dim R_x$.

(3)$\Rightarrow$(4). Let Ω fulfil (i)$\div$(iv). Condition (iv') holds in an evident manner. To show (v'), it is sufficient to notice that the manifold $\tilde{R}_x$ is equal to the leaf R_x, $x \in V$.

(4)$\Rightarrow$(1). Let us assume that a subset $\Omega \subset R$ has properties (i)$\div$(iii), (iv') and (v'). Take any abstract class L of R and $x \in L$. Via bijection $\gamma_x: L \to \tilde{R}_x$, $y \mapsto (x,y)$, we define some d.str. of a d.man. on L such that γ_x is a diff. By property (i) of $\tilde{R}_x$ (see lemma in [10]), we see the correctness of the definition of the d.man. L, while by property (ii) of $\tilde{R}_x$ - that the inclusion $L \hookrightarrow V = (L \xrightarrow{\gamma_x} \tilde{R}_x \hookrightarrow \{x\} \times V \xrightarrow{\cong} V)$ is an immersion. In view of assumption (v'), the family $\mathcal{F}$ of all arcwise connected components of all abstract classes of R is equal to the family of all connected components of all manifolds L obtained above. Let $x \in B \in \mathcal{F}$. From the definition of $\tilde{R}_x$ we see that $\Omega_x \subset \tilde{R}_x$, thus

$$T_xB = (pr_2)_{*(x,x)}[T_{(x,x)}\Omega_x]. \tag{4.6}$$

Put $T^{\alpha}\Omega := \mathrm{Ker}\,\alpha_*$, $\alpha := pr_1|\Omega$. It is a vector subbundle of order k of the tangent bundle $T\Omega$, which implies that $\hat{u}^*T^{\alpha}\Omega$ ($\hat{u}: V \to \Omega$, $x \mapsto (x,x)$) is a vector bundle of order k over V. Since $\varkappa: \hat{u}^*T^{\alpha}\Omega \to TV$, $v \mapsto (pr_2)_*v$, is a monomorphism of vector bundles (see the proof of theorem 2 in [10], part (2)$\Rightarrow$(1)), therefore, by (4.6), we see that $T\mathcal{F}$ is a vector subbundle of TV, which implies that $\mathcal{F}$ is a k-dim. foliation and, next, that each element $B\in\mathcal{F}$ is a leaf of V. Thus each d.man. L obtained above is a leaf, too, which gives that $\tilde{R}_x$ is a leaf of groupoid (1.4). Then, this last is a smooth groupoid. We have $A(R)_{|x}=T_{(x,x)}\tilde{R}_x=(T^{\alpha}\Omega)_{|(x,x)}\subset TR$. To finish the proof, we need notice the equality of d.s.'s $A(R)=T^{\alpha}\Omega_{|\Delta}$. Since $A(R)$ is (by definition) a proper d.subs. of TR and $T^{\alpha}\Omega_{|\Delta}$ - of $T\Omega$, it suffices to show that $T\Omega$ is a proper d.subs. of TR, but this results from the following lemma used to the situation when $\Omega\hookrightarrow R$. ∎

(4.7) LEMMA. If (M',C') is a d.subs. of (M,C), then the tangent d.s. $(T(M',C'),TC')$ is a d.subs. of the tangent d.s. $(T(M,C),TC)$, what is more, if for $U\in\tau_{C'}$, the equality $C'_U=C_U$ holds, then $(TC')_{\pi'^{-1}[U]}=(TC)_{\pi'^{-1}[U]}$ where $\pi':T(M',C')\longrightarrow M'$ is the projection. In particular, if (M',C') is a proper d.subs., then $(T(M',C'),TC')$ is such, too. ∎

Theorem (4.5) is the source of the notion of a nice structure (recall that the existence of the set Ω in that theorem is proved by using a nice covering of a foliation).

(4.8) DEFINITION. By a k-dim. nice structure of groupoid (1.1) we shall mean any subset $\Omega\subset\Phi$ such that

(i) $u[V]\subset\Omega$,

(ii) Ω is an n+k-dim. proper d.subman. of Φ,

(iii) $\alpha|\Omega:\Omega\longrightarrow V$ is a submersion,

(iv) $D_h[\Omega_y]\cap\Omega_x\subset\Omega_x:=(\alpha|\Omega)^{-1}(x)$, $h\in\Phi$, $x=\alpha h$, $y=\beta h$,

(v) connected components of the d.man. $\tilde{\Phi}_x$ (see lemma below) are equal to arcwise connected components of the subset $\alpha^{-1}(x)$ in Φ.

(4.9) LEMMA. If $\Omega\subset\Phi$ has properties (i)$\div$(iv) above, then, for each point $x\in V$, there exists exactly one k-dim. C^{∞}-manifold $\tilde{\Phi}_x$ with the support $\alpha^{-1}(x)$, such that, for each $h\in\alpha^{-1}(x)$,

(a) $D_h[\Omega_y]\subset\tilde{\Phi}_x$,

(b) $D_h|\Omega_y:\Omega_y\longrightarrow D_h[\Omega_y]\subset\tilde{\Phi}_x$ is a diff.

The manifolds $\tilde{\Phi}_x$ have the properties: (i) $D_h:\tilde{\Phi}_{\beta h}\longrightarrow\tilde{\Phi}_{\alpha h}$ is a diff., (ii) $\tilde{\Phi}_x$ are Hausdorff, (iii) $\tilde{\Phi}_x$ are d.subs. of Φ, (iv) if, in addition, this groupoid is a smooth groupoid with $\dim\Phi_x=k$ [$=k'$ and the man. $\tilde{\Phi}_x$ has a countable basis], then [$k=k'$ and] the leaf Φ_x is equal to $\tilde{\Phi}_x$, $x\in V$.

The proof is similar to that of the lemma from [10]. We only prove properties (ii) and (iv). To prove (ii), we first notice the continuity of the inclusion $j:\tilde{\Phi}_x\longrightarrow\Phi$ in some nbh of any point $h\in\alpha^{-1}(x)$:

$$j:\tilde{\Phi}_x\supset D_h[\Omega_{\beta h}]\xrightarrow[\approx]{D_h^{-1}}\Omega_{\beta h}\hookrightarrow(\Phi_{\beta h},C_{\Phi_{\beta h}})\xrightarrow{D_h}(\Phi,C).$$

Next, (ii) follows from the fact that (Φ,τ_C) is Hausdorff. To prove (iv), we notice that the identity map $\tilde{\Phi}_x \to \Phi_x$ is an immerse bijection between manifolds of the same dimensions [4,p. 101]. ∎

(4.10) THE SECOND OPEN PROBLEM. Are the manifolds $\tilde{\Phi}_x$ leaves of Φ ?

(4.11) DEFINITION. By a nice structure of smooth groupoid (1.1) we ahall mean any nice structure of this groupoid for which $\tilde{\Phi}_x = \Phi_x$, $x \in V$.

We see that $\Omega \subset \Phi$ is a nice structure of smooth groupoid (1.1) i f (i)÷(iii) as in (4.8), (iv′) Ω_x is an open subman. of Φ_x.

The notion of a groupoid with a nice structure is closely connected with the notion of "un morceau differentiable de groupoide" in the sense of J.Pradines [11].

(4.12) THEOREM. A smooth groupoid which has a nice structure is of Pradines type.

Proof. Let Ω be a nice structure of smooth groupoid (1.1), $k = \dim \Phi_x$. Then $T^{\alpha}\Omega := \mathrm{Ker}(\alpha|\Omega)_*$ is a vector subbundle of $T\Omega$ of order k. Thus $T^{\alpha}_0\Omega := T^{\alpha}\Omega_{|u[V]}$ is a vector bundle of order k over $u[V]$. Next, we see that $A(\Phi)_{|x} = T_{u_x}(\Phi_x) = T_{u_x}(\Omega_x) = (T^{\alpha}_0\Omega)_{|u_x}$, and that, by (4.7), $T^{\alpha}\Omega$ is a proper d.subs. of $T\Phi$. Thus $A(\Phi) = T^{\alpha}\Omega_{|u[V]}$ as d.s.'s, which implies that $A(\Phi)$ is a vector bundle. ∎

(4.13) THE THIRD OPEN PROBLEM. Has every Pradines-type groupoid a nice structure?

(4.14) REMARK. Let a family $\mathcal{F}$ of immerse connected submanifolds of V (covering V) be given. Take the groupoid (1.4) of the equivalence relation R whose family of abstract classes is equal to $\mathcal{F}$. We see that theorem (4.5) can be formulated as follows:

The conditions are equivalent:

(1) R is of Pradines type,

$\Updownarrow$ (some version of Frobenius' theorem [1,p.86])

(2) $\mathcal{F}$ is a foliation,

$\Updownarrow$ (some generalization of Godement's theorem [10])

(3) R has a nice structure. ∎

(4.15) EXAMPLE. Let G be a non-connected Lie group. G is, of course, a Lie groupoid with the one-point manifold of units. Each (open) nbh U of the neutral element $e \in G$ is a nice structure of this groupoid. It is well known that each connected nbh $U \ni e$ generates only the connected component of the element e in G. ∎

Let Ω be any subset of algebraic groupoid (1.1). By the groupoid generated by Ω we mean the smallest algebraic subgroupoid Ψ of Φ containing Ω. It is easy to see that Ψ consists of all finite products $h_n \cdot \ldots \cdot h_1$ only, where $h_i \in \Omega \cup \Omega^{-1}$ $(\Omega^{-1} = \{h^{-1};\ h \in \Omega\})$, $i \leq n$, $n \in N$.

(4.16) PROPOSITION. If Ω is a nice structure of groupoid (1.1) and Ψ is the subgroupoid generated by Ω, then the set $\Psi_x := \Psi \cap \alpha^{-1}(x)$

is an open-closed subset of $\tilde{\Phi}_x$, $x\in V$.

The very simply proof of this proposition is omitted. ■

(4.17) COROLLARY. Let $\mathcal{F}$ be any foliation of a d.man. V. Then each nice structure of $\mathcal{F}$, i.e. each nice structure of Pradines-type groupoid (1.4) of the equivalence relation determined by $\mathcal{F}$, generates the entire groupoid. ■

(4.18) DEFINITION. By a nice groupoid we shall mean a smooth groupoid Φ for which there exist nice structures Ω of Φ and Ω_0 of the groupoid of the equivalence relation

(4.19) $$R_\Phi = \{(x,y)\in V\times V: \bigvee_{h\in\Phi}(\alpha h=x,\ \beta h=y)\}\subset V\times V,$$

respectively, such that the mapping

(4.20) $$(\alpha,\beta)|\Omega : \Omega \longrightarrow \Omega_0$$

is a submersion.

For a nice groupoid Φ, the mapping $\bar{\beta}_x:\Phi_x \longrightarrow (R_\Phi)_x$, $h \mapsto (x,\beta h)$, is a submersion of leaves, $x\in V$.

The next theorem gives a class of nice groupoids. We need for this another notion from the theory of d.s.'s.

Let (M,C) and (N,D) be any d.s.'s. Smooth surjective mapping $f:(M,C) \longrightarrow (N,D)$ is called strong coregular if for some natural number n the following property holds:

- for each point $x\in M$ there exist nbh's $U\in\tau_C$ and $W\in\tau_D$ of x and f(x), respectively, and a diff. $\psi:(U,C_U) \longrightarrow (W,D_W)\times(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ such that $f|U = pr_1\circ\psi$.

(4.21) DEFINITION. Groupoid (1.1) is called strong coregular if

(4.22) $$(\alpha,\beta):\Phi \longrightarrow R_\Phi$$

is a strong coregular mapping (R_Φ - the proper d.subs. of $V\times V$).

(4.23) EXAMPLE. Groupoid (2.6) is strong coregular. Indeed, (α,β): $\Phi \longrightarrow V\times V$ is strong coregular because it is coregular between d.man.'s. The strong coregularity of Φ^R follows now from

(4.24) Lemma. If (M,C) and (N,D) are d.s.'s and $f:(M,C) \longrightarrow (N,D)$ is a strong coregular mapping then for each subset $N'\subset N$ the mapping $f|M': (M',C_{M'}) \longrightarrow (N',D_{N'})$, where $M'=f^{-1}[N']$, is strong coregular, too. What is more, if $(N',D_{N'})$ is a d.man., then $(M',C_{M'})$ is a d.man., too. ■

(4.25) THEOREM. Each strong coregular groupoid Φ for which the groupoid R_Φ is of Pradines type is a nice groupoid (in particular, is of Pradines type).

Proof. Let Ω_0 be any nice structure of R_Φ. Put $\Omega:=(\alpha,\beta)^{-1}[\Omega_0]$. By lemma (4.24), Ω is a proper d.subman. of Φ and mapping (4.20) is a submersion. Now, we show that Ω has properties (i)÷(iv) from definition (4.8) for $k=\dim\Omega-\dim V$. (i) and (ii) hold in an evident manner. (iii) results from the equality $\alpha|\Omega = pr_1\circ(\alpha,\beta)|\Omega$. To show (iv), take an arbitrary $h\in\Phi$ and put $x=\alpha h$, $y=\beta h$. From the assumption That Ω_o is a nice structure of R_Φ we have $D_{(x,y)}[\Omega_{oy}]\cap\Omega_{ox}\subset\Omega_{ox}$. Now, (iv) follows from the continuity of $(\alpha,\beta)|\Omega_x:\Omega_x\to\Omega_{ox}$ and the equality $D_h[\Omega_y]\cap\Omega_x=((\alpha,\beta)|\Omega_x)^{-1}[D_{(x,y)}[\Omega_{oy}]\cap\Omega_{ox}]$. Lemma (4.9) states that, for each point $x\in V$, there exists exactly one k-dim. d.man. $\tilde{\Phi}_x$ with the support $\alpha^{-1}(x)$, such that (a) and (b) from that lemma hold. Finally, it is sufficient to show that $\tilde{\Phi}_x$ is a leaf of Φ for each $x\in V$. First, we notice that the mapping $\bar{\beta}_x:\tilde{\Phi}_x\to(R_\Phi)_x$, $h\mapsto(x,\beta h)$, has the property:

- for each point $(x,y)\in(R_\Phi)_x$, there exists a nbh $W\subset(R_\Phi)_x$ of (x,y) such that (a) $(R_\Phi)_{x|W}$ is a proper d.subman. of the d.s. R_Φ, (b) $\tilde{\Phi}_{x|\beta_x^{-1}[W]}$ is a proper d.subman. of the d.s. Φ.

Indeed, for $(x,y)\in(R_\Phi)_x$, we can put $W:D_{(x,y)}[\Omega_{oy}]$. Now, the theorem follows from the lemma below. ■

(4.26) LEMMA. Let (M,C) and (N,D) be any d.s.'s. If $g:(M,C)\to(N,D)$ is a strong coregular mapping and (M',C') and (N',D') are d.subs.'s of (M,C) and (N,D), respectively, such that (1) (N',D') is a leaf of (N,D), (2) $M'=f^{-1}[N']$, (3) for each point $x\in M'$, there exists a nbh $U\in\tau_{D'}$ of $g(x)$ such that $D_U=D'_U$, $g^{-1}[U]\in\tau_{C'}$ and $C_{g^{-1}[U]}=C'_{g^{-1}[U]}$, then (M',C') is a leaf of (M,C).

The proof is identical with that for the analogous fact proved in theorem (2.5). ■

From this theorem we see that the strong coregular groupoid Φ^R from example (4.2)(3) is a nice groupoid.

Now, we explain the notion of the strong coregularity of smooth groupoids in the domain of differential groupoids.

(4.27) PROPOSITION. If (1.1) is a strong coregular differential groupoid with the connected space Φ, then equivalence relation (4.19) is regular in the sense of Godement [18,Ch.III.§12].

Proof. One should prove that (a) R_Φ is a proper d.subman. of $V\times V$, (b) $pr_1:R_\Phi\to V$ is a submersion. We see that (a) results from the lemma below and the assumption of the strong coregularity of (4.22), while (b) - from the equality $\alpha=pr_1\circ(\alpha,\beta)$. ■

The following lemma comes from the work by W.Waliszewski [24]. Now, we give a new short proof of this fact.

(4.28) LEMMA. If (M,C) and (N,D) are connected d.s.'s and $(M,C)\times(N,D)$ is a d.man., then (M,C) and (N,D) are d.man.'s, as well.

Proof. Take $x_o\in M$ and $y_o\in N$ and put $m:=\dim T_{x_o}(M,C)$, $n:=\dim T_{y_o}(N,D)$. Of course, $k:=m+n=\dim(M\times N,C\times D)$. There exist some nbh's $U\in\tau_C$ and $W\in\tau_D$ of x_o and y_o, respectively, and a diff. $\varphi:(U\times W,C_U\times D_W)\to(\Omega,C^\infty(\mathbb{R}^k)_\Omega)$ for some open subset $\Omega\subset\mathbb{R}^k$. We put $U_1:=\varphi[U\times\{y_o\}]$ and

$W_1 := \varphi[\{x_0\}\times W]$ and take the diff.'s

$$\varphi_1 = \varphi(\cdot, y_0):(U, C_U) \to (U_1, C^\infty(\mathbb{R}^k)_{U_1}),$$
$$\varphi_2 = \varphi(x_0, \cdot):(W, D_W) \to (W_1, C^\infty(\mathbb{R}^k)_{W_1}).$$

From the main theorem of paper [6] we infer - diminishing U and W, if necessary - that there exist some diff.'s

$$\psi_1:(U_1, C^\infty(\mathbb{R}^k)_{U_1}) \to (\Omega_1, C^\infty(\mathbb{R}^m)_{\Omega_1}),\ \Omega_1 \subset \mathbb{R}^m,$$
$$\psi_2:(W_1, C^\infty(\mathbb{R}^k)_{W_1}) \to (\Omega_2, C^\infty(\mathbb{R}^n)_{\Omega_2}),\ \Omega_2 \subset \mathbb{R}^n.$$

Hence the superposition

$$(\psi_1\circ\psi_2)\circ(\varphi_1\times\varphi_2)\cdot\varphi^{-1}:(\Omega, C^\infty(\mathbb{R}^k)_\Omega) \to (\Omega_1\times\Omega_2, C^\infty(\mathbb{R}^k)_{\Omega_1\times\Omega_2})$$

is a diffeomorphism. Therefore $\Omega_1\times\Omega_2$ is open in $\mathbb{R}^k$, so Ω_1 and Ω_2 are open in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. ∎

In view of remark (4.14), nice groupoids can be considered as some far-reaching generalization of foliations, and thus, the subsequent theorem (4.29) - as a generalization of Frobenius' theorem.

First, we recall the definition of a d.s. of the class $\mathfrak{D}_0$. Following P.G.Walczak [22], we denote by $\mathfrak{D}_0$ the largest class $\mathfrak{D}$ of d.s.'s, fulfilling the conditions:

(i) the class of d.man.'s is contained in $\mathfrak{D}$,

(ii) if $(M,C)\in\mathfrak{D}$, then $\dim T_x(M,C)<\infty$ for each $x\in M$,

(iii) if (M,C), $(M',C')\in\mathfrak{D}$, $f:(M,C)\to(M',C')$ is a smooth mapping and, for some $x\in M$, the differential $f_{*x}:T_x(M,C)\to T_{f(x)}(M',C')$ is an isomorphism, then there exists a nbh U of x open in τ_C such that $f|U:(U,C_U)\to(f[U],C'_{f[U]})$ is a diffeomorphism.

P.G.Walczak [23] proved the following

THEOREM. A d.s. (M,C) belongs to $\mathfrak{D}_0$ iff, for any $x\in M$, there exists its nbh $U\in\tau_C$ and a d.man. $\tilde{M}$ such that U is contained in the support of $\tilde{M}$, $\dim\tilde{M} = \dim T_x(M,C)$ and $C_U = C^\infty(\tilde{M})_U$.

The class $\mathfrak{D}_0$ is closed w.r.t. proper d.subs., i.e. if $(M,C)\in\mathfrak{D}_0$ and $A\subset M$, then $(A,C_A)\in\mathfrak{D}_0$ (see [23] and [6]). Thus, the space of the smooth groupoid Φ^R, constructed in theorem (2.5), is of the class $\mathfrak{D}_0$.

(4.29) THEOREM. A generalization of Frobenius' theorem.

Let (1.1) be any Pradines-type groupoid such that

(i) the d.s. Φ is of the class $\mathfrak{D}_0$, V is paracompact,

(ii) the groupoid R_Φ is of Pradines-type,

(iii) $\tilde{\beta}_x:\Phi_x\to(R_\Phi)_x$, $x\in V$, are submersions,

Then (1.1) is a nice groupoid.

Proof. Making use of a local extension of Φ to a d.man. and of some fact from the theory of fifferential equations, one can prove

(4.30) LEMMA. Let $x\in V$, and let $\xi_1,\dots,\xi_k\in\mathrm{Sec}\,A(\Phi)$ constitute a basis of $\mathrm{Sec}\,A(\Phi)$ over a nbh $W\subset V$ of x, such that (a) $\tilde{\beta}_*\circ\xi_1,\dots,\tilde{\beta}_*\circ\xi_n$, $n=\dim V$, constitute a basis of $\mathfrak{X}(V)$ over W, (b) $\tilde{\beta}_*\circ\xi_{n+1},\dots,\tilde{\beta}_*\circ\xi_k=0$.

Then there exists $\varepsilon > 0$, $K > 0$ and an open nbh $U \subset W$ of x, such that

$$\mathrm{Exp}: \overset{k}{\times}(-K,K)\times U \to \Phi, \quad (a,y)\mapsto \varphi_{y,a}(\varepsilon),$$

$$\exp: \overset{n}{\times}(-K,K)\times U \to R_{\Phi}\subset V\times V, \quad (a,y)\mapsto \psi_{y,a}(\varepsilon),$$

are diffeomorphisms onto their images Ω_U and Ω_{oU} (with d.str.'s induced from Φ and $V\times V$, respectively) and the diagram

$$\begin{array}{ccc} \overset{k}{\times}(-K,K)\times U & \longrightarrow & \Omega_U \\ \downarrow \mathrm{pr} & & \downarrow (\alpha,\beta) \\ \overset{n}{\times}(-K,K)\times U & \longrightarrow & \Omega_{oU} \end{array}$$

commutes, where $\varphi_{y,a}(\cdot)$ and $\psi_{y,a}(\cdot)$ denote the integral curves of the vector fields

$$\sum_{i=1}^{k} a^i \xi_i' \quad \text{and} \quad \sum_{i=1}^{n} a^i \tilde{\beta}_* \circ \xi_i,$$

passing through u_y and y, respevtively. ∎

According to that lemma we find a covering $\mathfrak{U}=\{U_s\}_{s\in S}$ and families of d.man.'s $\{\Omega_U;\ U\in\mathfrak{U}\}$, $\{\Omega_{oU};\ U\in\mathfrak{U}\}$. Let $\mathfrak{U}'$ be any open covering of V su h that $\overline{U'}$ are compact for $U'\in\mathfrak{U}'$ and $\overline{\mathfrak{U}}':=\{\overline{U'};\ U'\in\mathfrak{U}'\}$is a refinement of a covering $\mathfrak{U}$.

Next, making use of strong paracompactness of V we refine starlikely $\mathfrak{U}'$ to some open starlike finite covering $\mathfrak{W}=\{W_t\}_{t\in T}$. Let

$$\mathrm{St}(W_t;\mathfrak{W}) = \bigcup_{i=1}^{m} W_{t_i}\subset U'_{s'(t)}\subset \overline{U'}_{s'(t)}\subset U_{s(t)}, \quad m=m(t),\ t\in T.$$

In this connection,

$$W_t \subset U_{s(t_1)}\cap\ldots\cap U_{s(t_m)},$$

and if $W_t\cap W_{t'}\neq\emptyset$ then $W_t\cup W_{t'}\subset U_{s(t)}\cap U_{s(t')}$. For each $t\in T$ we take the basis of $\mathrm{Sec}\,A(\Phi)$ over W_t consisting of all restrictions to W_t of all elements of the basis over $U_{s(t)}$ considered in lemma (4.30). Making several times use of the same fact from differential equations (this time to the manifolds Ω_{U_s}) we obtain:

For each $t\in T$ there is $K_t>0$ such that the mappings (defined as above) $\mathrm{Exp}: \overset{k}{\times}(-K_t,K_t)\times W_t \to \Phi$ and $\exp: \overset{n}{\times}(-K_t,K_t)\times W_t \to R_{\Phi}\subset V\times V$ are diffeomorphisms onto their images Ω_t, Ω_{ot}, $(\alpha,\beta):\Omega_t\to\Omega_{ot}$ is a submersion, $\Omega_t\subset\Omega_{U_{s(t_1)}}\cap\ldots\cap\Omega_{U_{s(t_m)}}$ and $\Omega_{ot}\subset\Omega_{oU_{s(t_1)}}\cap\ldots\cap\Omega_{oU_{s(t_m)}}$. Therefore, if $W_t\cap W_{t'}\neq\emptyset$ then

$$\Omega_t\cup\Omega_{t'}\subset\Omega_{U_{s(t)}}.$$

From the construction, we see that Ω_t and $\Omega_{t'}$ are k-dim. submanifolds of the k-dim d.man. $\Omega_{U_{s(t)}}$, so they are open in the last manifold. This implies that $\Omega_t \subset \Omega_t\cup\Omega_{t'}$ and $\Omega_{ot}\subset\Omega_{ot}\cup\Omega_{ot'}$. Then

$$\Omega_t\subset\Omega := \bigcup_t \Omega_t \quad\text{and}\quad \Omega_{ot}\subset\Omega_o := \bigcup_t \Omega_{ot}.$$

Hence Ω and Ω_o satisfy the required conditions. ∎

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