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COMPLEX COHOMOLOGY AUTOMORPHISMS OF COMPACT HOMOGENEOUS SPACES OF POSITIVE EULER CHARACTERISTIC

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Introduction

Let G be a compact connected semisimple Lie group and let K be a proper closed connected subgroup of the same rank. Consider a common maximal torus and denote by V its Lie algebra. One then has a pair of root systems, $R=(R_K \subset R_G \subset V)$, and a pair of Weyl groups, $(W_K \subset W_G \subset GL(V\otimes \mathbb{F}))$, $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, which naturally act on the polynomial graded algebra on $V\otimes \mathbb{F}$ (graded by $\deg(V\otimes\mathbb{F})^*=1$), giving thus rise to a pair of graded subalgebras of invariants, $(I_G(\mathbb{F})\subset I_K(\mathbb{F}))$. One knows that $H^*(G/K;\mathbb{F}) = I_K(\mathbb{F})/I_K(\mathbb{F}).I_G^+(\mathbb{F})$ (as graded algebras, provided the degrees of the right hand side are doubled). Consider next the normalizers of the Weyl groups, $N_G(\mathbb{F})=N_{GL(V\otimes\mathbb{F})}(W_G)$ (and similarly for K) and the group $N(\mathbb{F})=N_G(\mathbb{F})\cap N_K(\mathbb{F})$, which naturally acts on the polynomial algebra on $V\otimes\mathbb{F}$, preserving the invariant subalgebras and thus giving rise to a group morphism $p:N(\mathbb{F})\longrightarrow \mathrm{Aut}H^*(G/K;\mathbb{F})$, whose image was considered in [9] under the name of "cohomology automorphisms of Lie type".

This paper is devoted to the study of $\mathrm{Aut}H^*(G/K;\mathbb{C})$, centered around the general question: are all cohomology automorphisms of Lie type? This question makes sense for any characteristic zero field coefficients $\mathbb{F}$ (see [9]); if K=maximal torus, then the answer is yes, for $\mathbb{F}=\mathbb{Q},\mathbb{R}$ [8]. Our first result here establishes the same answer for $\mathbb{F}=\mathbb{C}$ and gives a precise description of $\mathrm{Aut}H^*(G/T;\mathbb{C})$, T = maximal torus. Consider the orthogonal decomposition $V=\oplus V^i$ (corresponding to the infinitesimal splitting of G as a product of simple groups) and denote by $D(\mathbb{F})\subset GL(V\otimes\mathbb{F})$ the subgroup of elements which act as scalars of $\mathbb{F}^*$ on each $V^i\otimes\mathbb{F}$ ($\mathbb{F}=\mathbb{R},\mathbb{C}$).

Theorem 1. p is an isomorphism $N(\mathbb{C})\xrightarrow{\sim}\mathrm{Aut}H^*(G/T;\mathbb{C})$ *and* $N(\mathbb{C})=D(\mathbb{C}).N(\mathbb{R})$.

For a complete description of $N(\mathbb{R})$, see [8].

If G=SU(n), then the conjecture of [4,7] on $\mathrm{Aut}H^*(G/K;\mathbb{Q})$ is equivalent to the fact that all $\mathbb{Q}$-cohomology automorphisms are of Lie type ([9]), and was verified in many particular cases, by several authors. On the other hand, there are examples where not all $\mathbb{F}$-cohomology automorphisms are of Lie type (see [9] for $\mathbb{F}=\mathbb{Q},\mathbb{R}$, and the example given in the next section, for $\mathbb{F}=\mathbb{C}$), therefore a more reasonable question would be: <u>when</u> are all $\mathbb{F}$-cohomology automorphisms of Lie type?

Our next result provides an equivalent formulation of this property ($\mathbb{F}=\mathbb{R},\mathbb{C}$). Consider the graded $\mathbb{F}$-vector space $Q_G=I_G^+/I_G^+\cdot I_G^+$ (similarly for K) and the linear degree zero map $Q_i:Q_G\to Q_K$ induced by the inclusion $i:I_G\subset I_K$; denote its kernel by h^o, its cokernel by h^e and set $h=h^o\oplus h^e$. Since plainly Q_i commutes with the obvious actions of N on Q_G and Q_K, we may consider <u>the odd, even and total dual homotopy representations</u> of N in $GL(h^o)$, $GL(h^e)$ and $GL(h)$, to be denoted in the sequel by r_L^o, r_L^e and r_L. Rational homotopy theory [10] identifies h^o, h^e and h with the graded spaces of odd-dimensional, even-dimensional, respectively all multiplicative generators of the $\mathbb{F}$-minimal model of G/K (and consequently with $(\pi_{odd}(G/K)\otimes\mathbb{F})^*$, $(\pi_{even}(G/K)\otimes\mathbb{F})^*$, respectively $(\pi_*(G/K)\otimes\mathbb{F})^*$, which explains our terminology). Since G/K is formal, $\mathrm{Aut}H^*(G/K)$ acts (up to algebraic homotopy) on the $\mathbb{F}$-minimal model, thus inducing (genuine) representations in $GL(h^o)$, $GL(h^e)$ and $GL(h)$, to be denoted by r_H^o, r_H^e and r_H (the precise construction of these <u>dual homotopy representations</u> of $\mathrm{Aut}H^*(G/K)$ is given in Section 2).

<u>*Theorem 2.*</u> *Suppose that the unipotent radical (see e.g. [6]) of the linear algebraic group* $\mathrm{Aut}H^*(G/K;\mathbb{C})$ *is trivial. Then* p *is onto if and only if* $r_L^e(N(\mathbb{F}))=r_H^e(\mathrm{Aut}H^*(G/K;\mathbb{F}))$, $\mathbb{F}=\mathbb{C},\mathbb{R}$.

We remark that the assumption on the unipotent radical is always fulfilled if G is simple (by the main result of [11], which states that the identitycomponent of $\mathrm{Aut}H^*(G/K;\mathbb{C})$ is a 1-dimensional algebraic torus). On the other hand r_L turns out to be quite manageable (see Sections 2,3).

<u>*Theorem 3.*</u> *If* G *is simple and* W_K *is a normal subgroup of* W_G*, then all complex cohomology automorphisms of* G/K *are of Lie type.*

We should point out that the statement above is false for real coefficients (see Section 3). Needless to say, complexification is often a useful device; in our case, it turns out to be rather obligatory, which finally reformulates our main question as: when are all <u>complex</u> cohomology automorphisms of Lie type?

1. Compact Lie groups modulo maximal tori

We begin by making some preliminary remarks, on the way of proving Theorem 1.As a notational simplifying convention, we are going to suppress the subscript G (recalling that, when K=T, R_K is void and W_K is trivial). Denoting, for $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, by $A(\mathbb{F})$ the subgroup of $GL(V\otimes\mathbb{F})$ consisting of those elements whose natural action on $\mathbb{F}[V\otimes\mathbb{F}]$ preserves the ideal generated by $I^+(\mathbb{F})$, notice that $N(\mathbb{F})\subset A(\mathbb{F})$, that there is a natural group morphism :

$$p : A(\mathbb{F})\longrightarrow \mathrm{Aut}H^*(G/T;\mathbb{F})$$

which extends our p in the theorem, and which is an isomorphism ([8], Prop.2.1); [8] also gives that $A(\mathbb{R})=N(\mathbb{R})$. Complexification induces inclusions $A(\mathbb{R})\subset A(\mathbb{C})$ and $N(\mathbb{R})\subset N(\mathbb{C})$; to be more precise $A(\mathbb{R})=A(\mathbb{C})\cap GL(V)$ and $N(\mathbb{R})=N(\mathbb{C})\cap GL(V)$.

We claim now that it will be enough to show that

$$(1) \qquad A(\mathbb{C})\subset A(\mathbb{R}).D(\mathbb{C})$$

Indeed, knowing this we immediately deduce that $A(\mathbb{C})=N(\mathbb{C})$, hence our first assertion of the theorem, and next that $N(\mathbb{C})=N(\mathbb{R}).D(\mathbb{C})$. The other assertion is a consequence of the fact that $N(\mathbb{R}).D(\mathbb{C})=$ $=D(\mathbb{C}).N(\mathbb{R})$, which in turn follows from the fact (proved in [8]) that the action of $N(\mathbb{R})$ on V permutes the decomposition $V=\oplus V^i$.

Choose a system of simple roots for R, $S=\coprod S^i$ $(S^i\subset V^i)$, and consider the associated positive roots, $R_+\subset R$.

1.1. Lemma. *For any* $g\in A(\mathbb{C})$ *and for any* $a\in R$ *there exist (uniquely)* $t_a\in\mathbb{C}^*$ *and* $q_a\in R_+$ *such that* $g(a)=t_a.q_a$.

Proof. Uniqueness is clear. The existence proof is essentially the proof of Theorem 1.1[8]. Denote by n the number of positive roots, recall that $\dim(G/T)=2n$ and consider the nonzero degree n homogeneous polynomial function on $(V\otimes\mathbb{C})^*$ defined by

$$(2) \qquad J(x)=\langle x^n,[G/T]\rangle \ , \quad x\in(V\otimes\mathbb{C})^*$$

Also consider the nonzero degree n homogeneous polynomial $J_o=\prod_{a\in R_+} L_a$, where $L_a(x)=x(a)$, $x\in(V\otimes\mathbb{C})^*$. One infers from [1] that J is a nonzero complex multiple of J_o. If $g\in A(\mathbb{C})$ then clearly $J\circ g^*$ is a nonzero multiple of J, hence g^* permutes the irreducible factors of J_o (up to nonzero complex scalars), i.e. given any $a\in R_+$ there exist $t\in\mathbb{C}^*$ and $b\in R_+$ such that $L_a\circ g^*=t.L_b$, that is $L_{g(a)}=L_{t.b}$, whence $g(a)=t.b$, which gives the lemma.

1.2. Lemma. *Fix* $g\in A(\mathbb{C})$ *and keep the notations of the previous* *lemma. If* a,b$\in$R *and* a+b$\in$R *then* $t_a \in \mathbb{R}^*$ *if and only if* $t_b \in \mathbb{R}^*$.

Proof. Suppose that $t_a \in \mathbb{R}^*$, but $t_b \notin \mathbb{R}^*$ and write that $g(a)=t_a.q_a$, $g(b)=t_b.q_b$, $g(a+b)=t_{a+b}.q_{a+b}=t_a.q_a+t_b.q_b$. Equating the imaginary parts of the last equality, we find out that q_{a+b} and q_b are proportional, which implies that the roots a+b and b are proportional (over $\mathbb{R}$), whence a+b=$\pm$b, which is absurd.

1.3. End of proof of Theorem 1. Pick a simple root $a_i \in S^i$, for any i. Given $g \in A(\mathbb{C})$, write $g(a_i)=t_i.b_i$, with $t_i \in \mathbb{C}^*$ and $b_i \in R_+$ (by Lemma 1.1). Define $d \in D(\mathbb{C})$ by $d=\mathrm{diag}(t_i^{-1})$ and notice that $gd(a_i)=b_i \in \mathbb{R}^*.R_+$, for any i. Given any $c_i \in S^i$, choose a path connecting c_i to a_i in the Coxeter graph, repeatedly apply Lemma 1.2 to gd and conclude that $gd(S^i) \subset \mathbb{R}^*.R_+ \subset V$, for any i. Since S is known to generate V as an $\mathbb{R}$-vector space [1], we infer that $gd \in GL(V)$, hence $gd \in A(\mathbb{R})$, which proves the desired inclusion (1) and thus finishes the proof of Theorem 1.

1.4. *Corollary. For a general maximal rank subgroup* K$\subset$G *we have* $N(\mathbb{C})=D(\mathbb{C}).N(\mathbb{R})$.

Proof. We have just seen that $N_G(\mathbb{C}) \subset D(\mathbb{C}).GL(V)$, hence $N(\mathbb{C}) \subset D(\mathbb{C}).N(\mathbb{R})$ (since $D(\mathbb{F}) \subset N(\mathbb{F})$, due to the fact that $W_G=xW_G^i$, with $W_G^i \subset GL(V)$, and similarly for W_K). The other inclusion is clear.

1.5. Example. Consider $U(3) \subset SO(7)$ (Example 6.9 of [9]). We have noticed there that $p(N(\mathbb{R}))$ consists of grading $\mathbb{R}$-automorphisms (i.e. those which act on H^{2j} as $t^j.\mathrm{id}$, for some $t \in \mathbb{R}^*$) and exhibited an $\mathbb{F}$-cohomology automorphism ($\mathbb{F}=\mathbb{R},\mathbb{C}$) which is not a grading $\mathbb{F}$-automorphism. By the previous corollary $p(N(\mathbb{C}))=p(\mathbb{C}^*).p(N(\mathbb{R}))$ will again consist only of grading $\mathbb{C}$-automorphisms, which shows that not all automorphisms of $H^*(SO(7)/U(3);\mathbb{C})$ are of Lie type.

2. The dual homotopy representations

We start by constructing the dual homotopy representation r_H of $\mathrm{Aut}H^*(G/K;\mathbb{F})$ in GL(h) ($\mathbb{F}=\mathbb{R},\mathbb{C}$). In order to do this, we begin by recalling the classical construction of a free dga model of $H^*(G/K)$. Set $M=I_K \otimes \Lambda \bar{Q}_G$, where $\bar{Q}_G$ is the desuspension of the graded $\mathbb{F}$-vector space Q_G and the degrees of I_K and Q_G are defined by doubling the usual degrees of $\mathbb{F}[V \otimes \mathbb{F}]$. A section of the canonical projection $I_G^+ \to Q_G$ defines a degree 1 linear map $d:\bar{Q}_G \to I_K$, which extends to a differential $d:M\rightarrow M$ (by setting $d(I_K)=0$). A dga map $m_o:(M,d)\rightarrow(H^*(G/K),0)$ is defined by $m_o|I_K$=canonical projection and $m_o(\bar{Q}_G)=0$; it induces an isomorphism in cohomology. Given any dga(A,d), consider the graded vector space $A^+/A^+\cdot A^+$, denote by Q_d (following [5]) the induced differential and define $Q^*(A,d)=H^*(A^+/A^+\cdot A^+,Q_d)$, noticing that this construction is natural with respect to dga maps. In our case $Q^*(M,d)$ is independent of the choice made in the construction of d; more precisely $Q^{2n}(M,d)=(h^e)^n$ and $Q^{2n-1}(M,d)=(h^o)^n$, for any n (with the notations of the Introduction). Given $g\in\mathrm{Aut}H^*(G/K)$, the general theory (cf.[10]) guarantees the existence of a dga map $\bar{g}:M\rightarrow M$ (which is unique up to algebraic homotopy) with the property that $m_o\bar{g}\simeq gm_o$. It follows that $Q(\bar{g}):Q(M,d)\rightarrow Q(M,d)$ depends only on g, and we construct the dual homotopy representation r_H by setting $r_H(g)=Q(\bar{g})\in GL(h)$. As far as the dependence on $\mathbb{F}$ is concerned, we just have to notice that $H^*(G/K;\mathbb{C})=H^*(G/K;\mathbb{R})\otimes\mathbb{C}$ (which embeds $\mathrm{Aut}H^*(G/K;\mathbb{R})$ into $\mathrm{Aut}H^*(G/K;\mathbb{C})$ by complexification), that $h(\mathbb{C})=h(\mathbb{R})\otimes\mathbb{C}$ (embedding $GL(h(\mathbb{R}))$ into $GL(h(\mathbb{C})))$, and that (choosing $d(\mathbb{R})\otimes\mathbb{C}$ as $d(\mathbb{C})$) $r_H(\mathbb{C})$ restricts to $r_H(\mathbb{R})$.

This construction is "geometric", from the point of view of rational homotopy theory (recall that the homotopy classes of selfmaps of the rationalization of G/K are in natural bijection with the graded algebra endomorphisms of $H^*(G/K;\mathbb{Q})$, see [3]). A second (simpler) construction will better suit our purpose here. Abbreviate $H^*(G/K)$ to H^* and set $r(g)=Q(g)\in GL(Q(H^*,0))$, for any $g\in\mathrm{Aut}H^*$. It is immediate to see that $Q(H^*)=h^e$ and that $r(g)=r^e_H(g)$. (For the second assertion, recall that $H^*m_o=\mathrm{id}$, which shows that $r(g)=Q(H^*\bar{g})$, next that there is an obvious degree zero map $Q(H^*A,0)\rightarrow Q(A,d)$, natural in the dga$(A,d)$ and which equals the identity when d=0, apply this naturality property to $m_o:(M,d)\rightarrow(H^*M,0)$ and deduce that $Q^{even}(M,d)=$ $=Q(H^*M,0)$).

We move now to the proof of Theorem 2. The first step is the following self-evident remark (in our second setting)

(1) $$r^e_H\circ p=r^e_L$$

(We point out that it is not difficult to see that the same holds for r^o). It follows that without any other assumption we always have $r^e_L(N(\mathbb{F}))\subset r^e_H(\mathrm{Aut}H^*(G/K;\mathbb{F}))$ (and similarly for r^o) and equality must hold if p is onto.

In order to prove the converse we invoke the following general fact: if H is a connected finitely generated commutative graded algebra then Aut(H) is a linear algebraic group and ker r (where $r(g)=Q(g)$, as above) is a unipotent subgroup of Aut(H). Proof:

(sketch):set $Q(H)=Q$ and use a section of $H^{+}\twoheadrightarrow Q(H)$ in order to write down a finitely generated presentation of H

(2) $$0 \to J \to \Lambda Q \xrightarrow{P} H \to 0$$

which exhibits Aut(H) as a quotient of the subgroup of Aut(ΛQ) consisting of elements which leave J invariant. If $r(g)=id$, $g\in Aut(H)$, then g comes from some $f\in Aut(\Lambda Q)$ (leaving J invariant) and $r(f)=id$ (since $Q(P)$ is a isomorphism); but then clearly f must be unipotent, hence g is also unipotent.

If the unipotent radical of $AutH^*(G/K;\mathbb{C})$ is trivial, then r^e_H must be monic (for $\mathbb{F}=\mathbb{C}$ and consequently also for $\mathbb{F}=\mathbb{R}$). Given the equality (1), $r^e_L(N(\mathbb{F}))=r^e_H(AutH^*(G/K;\mathbb{F}))$ forces then p to be onto. Theorem 2 is proved.

We close this section by saying a little more about r_L. First of all, we have natural representations r_G (of N_G in $GL(Q_G)$) and r_K(of N_K in $GL(Q_K)$), whose restrictions to N fit into an exact sequence

(3) $$0 \to h^o \to Q_G \xrightarrow{Q_i} Q_K \to h^e \to 0$$

The main result (which is of great help in making explicit computations, see e.g. next section) is the following.

2.1. Proposition. *If F is a finite subgroup of $N_G(\mathbb{R})$ which leaves some W_G-chamber invariant, then Q_G and V are isomorphic as F-modules. The same also holds for K.*

Proof. Implicit in the proof of Lemma 3.2 [8], when G is semisimple. We briefly discuss the extra-arguments needed for the general case (K might not be semisimple!). We are going to supress the subscript G and recall from [9] that one has an orthogonal decomposition $V=V^W\oplus V_W$ (with V^W=fixed points of W and $V_W=\mathbb{R}$-span(R)) and compatible splittings $W=\{1\}\times W$ and $N_{GL(V)}(W)=GL(V^W)\times N^{(W)}_{GL(V_W)}$, where $R\subset V_W$ is the root system of a semisimple group. These splittings induce F-module splittings $V=V^W\oplus V_W$ and $Q=V^W\oplus Q_{SS}$, where the F-module structures on V^W are the same, and we are thus reduced to the already settled semisimple case.

This can be used for example in the following way: since $r_K(v)=id$ and $r_G(v)=id$, for any $v\in W_K$, we may work with N/W_K instead of N, fix a pair of Weyl chambers, $C_G\subset C_K$, denote by [n] the class of $n\in N$ mod W_K and (remembering that the elements of N act on W_G and W_K-chambers, see [8 , 9]) we may always suppose that n has been normalized, i.e. $n(C_K)=C_K$, cf. [1] (here and in the following statement

$\mathbb{F}=\mathbb{R}$). By [1] again, there is a unique $u\in W_G$ such that $n(C_G)=u(C_G)$.

2.2. *Corollary. Suppose* $n\in N(\mathbb{R})$ *is normalized and of finite order. Then the characteristic polynomials of* $r_K(n)$ *and* n *(respectively of* $r_G(n)$ *and* $u^{-1}n$ *) coincide.*

3. Complex versus real coefficients. Examples

This section is devoted to the proof of Theorem 3. We are dealing in fact with a root system pair, $R=(R_K\subset R_G\subset V)$, where R_G is supposed to be normalized (i.e. $V=\mathbb{R}$-span(R_G)) and irreducible, and R_K is a proper closed ([2 , 1]) subsystem. We may also suppose that R_K is nonvoid (otherwise we are done, by Theorem 1).

3.1. *Lemma. Under the above assumptions,* W_K *is a normal subgroup of* W_G *if and only if* R_G *has two root lengths and* R_K*=long roots of* R_G.

Proof. Given an arbitrary root system R, it is immediate to see that the roots of a given length ℓ form a subsystem R_ℓ (eventually void, or equal to R). If $a,b\in R_\ell$ and $a+b\in R$, we compute the square of the length of a+b as $(a+b,a+b)=\ell^2(2+\langle a,b\rangle)\geq\ell^2$, since the Cartan integer $\langle a,b\rangle$ must be equal to 0 or ± 1, see [1]. This shows that the roots of maximal length of R form a closed subsystem (which is nonvoid and proper if R has more than one root length). On the other hand the Weyl group $W(R_\ell)$ is always normal in W(R). Slightly more generally, given an arbitrary root system $R\subset V$ and an isometry $f\in O(V)$, f normalizes W(R) if and only if $f(R)\subset R$ (since it is enough to check f on the generators of W(R), since $fS_af^{-1}=S_{f(a)}$, $a\in R$ - where S_v denotes the symmetry with respect to the hyperplane orthogonal to $v\in V$ - and since the only symmetries in W(R) are those of the form S_a, $a\in R$ - see [1]). Half of our statement is thus verified. Finally assume that W_K is normal in W_G. As we have seen, this means that $W_G(R_K)\subset R_K$. Since, as it is well-known [1] all roots of the same length of an irreducible root system are conjugate under the action of its Weyl group, this leaves us with two possibilities (R_K being proper and nonvoid): either $R_K=(R_G)_{long}$ or $R_K=(R_G)_{short}$ (and of course forces R_G to have two root lengths). It can be easily checked (e.g. by direct inspection) that the short roots of R_G do not form a closed subsystem, whence the lemma.

3.2. Proof of Theorem 3

We are going to check separately the various cases (for both $\mathbb{F}=\mathbb{C}$ and $\mathbb{R}$). The classification [1] says that R_G mus be $B_\ell(\ell\geq 2)$,

$C_\ell (\ell \geq 3)$, F_4 or G_2, and R_K must respectively be D_ℓ, A_1^ℓ, D_4 or A_2. In all cases $V=\mathbb{R}^\ell$, with standard basis $\{e_1,\ldots,e_\ell\}$, coordinates $(X_1,\ldots,X_\ell)$ and euclidean metric, R_G will be in standard form, as in [1], and with a standard choice of simple roots.

Given a commutative graded algebra A, graded by even-dimensional degrees, and a positive integer m, we define an algebra of the same kind, denoted by $m\cdot A$, by simply multiplying by m the degrees of A. Notice that A and $m\cdot A$ have the same group of automorphisms. The reason for waisting time with such a definition is that the proof of our theorem aposteriori gives the following curious result: if G is simple and W_K is normal then $H^*(G/K;\mathbb{F})=mH^*(U(n)/T;\mathbb{F})$ for some m and n; we have no apriori explanation of this phenomenon. Any way, in what follows it is good to bear in mind that $\operatorname{Aut}H^*(U(n)/T;\mathbb{F})$ is generated by $\mathbb{F}^*$ (which acts by grading $\mathbb{F}$-automorphisms) and the symmetric group S_n (which naturally acts by permutation of coordinates in $\mathbb{R}^n$)-see [8], and Theorem 1 of this paper. As far as $\operatorname{Aut}(m.H^*(U(n)/T;\mathbb{F}))$ is concerned, there is one more point: given $t\in\mathbb{F}^*$, it acts on $m.H^*$ as $gr_m(t)=t^i.\mathrm{id}$ on $(m.H^*)^{2mi}=H^{2i}$; for m=1, this is an usual grading $\mathbb{F}$-automorphism; if $\mathbb{F}=\mathbb{C}$, or $\mathbb{F}=\mathbb{R}$ and either m is odd or $t\in\mathbb{R}^+$, then $gr_m(t)=gr_1(t^{1/m})$ and we still get usual $\mathbb{F}$-grading automorphisms (which are of Lie type). On the other hand, if $\mathbb{F}=\mathbb{R}$ and m is even, then $gr_m(-1)$ is not an $\mathbb{R}$-grading automorphism, and this explains the different behaviour of real coefficients, see the remark below. In what follows we will ckeck that always in our list $S_n\subset p(N(\mathbb{R}))$, for $n>2$ (remember that $\operatorname{Aut}H^*(U(2)/T;\mathbb{F})=\mathbb{F}^*$), thus settling the case $\mathbb{F}=\mathbb{C}$ and finishing the proof of Theorem 3, and also check that $gr_m(-1)\in p(N(\mathbb{R}))$, if $R_G=B_\ell$ or G_2. The discussion of real coefficients will be completed by the next remark, namely by showing that $gr_m(-1)\notin p(N(\mathbb{R}))$ if $R_G=C_\ell$ or F_4.

(1) $R=(D_\ell\subset B_\ell)$. $H^*(G/K;\mathbb{F})=H^*(S^{2\ell};\mathbb{F})=\ell.H^*(U(2)/T;\mathbb{F})$.

In terms of Weyl groups invariants $H^*(G/K;\mathbb{F})$ is generated by the Euler class $e=X_1\ldots X_\ell$, with the relation $e^2=0$. Consider the linear transformation $w(X_1,\ldots,X_\ell)=(-X_1,\ldots,X_\ell)$, $w\in W_G\subset N(\mathbb{R})$ and notice that $p(w)=gr_\ell(-1)$.

(2) $R=(A_2\subset G_2)$. As it is well-known, $H^*(G_2/SU(3);\mathbb{F})= =H^*(S^6;\mathbb{F})=3.H^*(U(2)/T;\mathbb{F})$. Moreover $gr_3(-1)=gr_1(-1)\in p(\mathbb{R}^*)$.

By the above discussion, in these two cases all $\mathbb{F}$-cohomology automorphisms are of Lie type, for both $\mathbb{F}=\mathbb{C}$ and $\mathbb{R}$.

(3) $R=(A_1^\ell\subset C_\ell)$. It is equally well-known that $H^*(Sp(\ell)/Sp(1)^\ell;\mathbb{F}) =2.H^*(U(\ell)/T;\mathbb{F})=\mathbb{F}[X_1^2,\ldots,X_\ell^2]/(p_1,\ldots,p_\ell)$, where p_j is the j-th elementary symmetric function of $X_1^2,\ldots,X_\ell^2$, and that $S_\ell\subset W_G\subset N(\mathbb{R})$ and acts

by permutation of coordinates ([1]).

(4) $R=(D_4 \subset F_4)$. Since W_K is normal in W_G, we know that $W_G \subset \mathrm{Aut}(R_K)$ (the group of automorphisms of the root system R_K, [1]), see the proof of Lemma 3.1. We also know that $\mathrm{Aut}(R_K)=\mathrm{Dgraut}(S_K) \ltimes W_K$, where S_K are simple roots of R_K, $\mathrm{Dgraut}(S_K)$ denotes the automorphism group of the associated Dynkin diagram, whose elements leave the W_K-chamber C_K invariant, see [1].

In our case, $\mathrm{Dgraut}\ (S_K)=S_3$.
$W_G=\mathrm{Aut}(R_K)$, by a cardinality argument, see [1]. It follows that $I_G=(I_K)^{S_3}$ (the invariants of S_3 in I_K) and that Proposition 2.1 is available, for $S_3 \subset N_K(\mathbb{R})$. As a graded vector space, it is well-known that $Q_K^*=Q^2 \oplus Q^6 \oplus Q^4$, with $\dim Q^2=\dim Q^6=1$ and $\dim Q^4=2$. We also know that S_3 acts trivially on Q^2, since $S_3 \subset O(V)$ and Q^2 is generated by the W-invariant metric on V. On the other hand $g \in S_3$ is known to act on V via the permutation of the $\mathbb{R}$-basis of V given by the simple roots a_1,a_2,a_3,a_4 of R_K which fixes a_2 and coincides with g on the remaining roots, hence V is isomorphic as an S_3-module with $U \oplus V(A_2)$, where U is 2-dimensional and trivial and $V(A_2)$ is the 2-dimensional irreducible defining representation of the Weyl group $W(A_2)=S_3$. Using Proposition 2.1 we deduce that $I_G=\mathbb{F}\left[(Q^2 \oplus Q^6) \otimes \mathbb{F}\right] \otimes \mathbb{F}\left[Q^4 \otimes \mathbb{F}\right]^{W(A_2)}$, where $Q^4=V(A_2)$, hence $H^*(G/K;\mathbb{F})=4.H^*(U(3)/T;\mathbb{F})$. Finally $S_3 \subset W_G \subset N(\mathbb{R})$, by construction.

3.3. <u>Remark</u>. If $R_G=C_\ell\ (\ell \geq 3)$ or $R_G=F_4$ then not all <u>real</u> cohomology automorphisms are of Lie type. In the first case notice first that $N_G(\mathbb{R}) = \mathbb{R}^+ \times W_G$ ([8]), hence $N(\mathbb{R}) = \mathbb{R}^+ \times W_G$. Given the concrete description of $H^*(G/K;\mathbb{R})$ we see that the assumption $gr_2(-1) \in p(N(\mathbb{R}))$ would imply that $-1 \in \mathbb{R}^+.W(A_{\ell-1})$, hence $-1 \in W(A_{\ell-1})$, which is absurd. If $R_G=F_4$ the same argument shows that $gr_4(-1) \notin p(\mathbb{R}^+ \times W_G)$. But in this case too we have $N(\mathbb{R}) = \mathbb{R}^+ \times W_G$. This can be as follows: the split exact sequence which describes $N_G(\mathbb{R})$ [8]

$$1 \to \mathbb{R}^+ \times W_G \to N_G(\mathbb{R}) \underset{\sigma}{\rightleftarrows} \mathrm{Graphaut}(S_G) \longrightarrow 1$$

(in which $\mathrm{Graphaut}(S_G)=\mathbb{Z}_2$, with nontrivial element say g) restricts to an exact sequence (see [9])

$$1 \to \mathbb{R}^+ \times W_G \longrightarrow N(\mathbb{R}) \longrightarrow \mathrm{Graphaut}(C) \longrightarrow 1$$

If $g \in \mathrm{Graphaut}(C)$ then necessarily $\sigma(g) \in N_K(\mathbb{R})$. But we know (cf. [9], 6.8) that for any long root $b \in F_4$, $\sigma(g) s_b \sigma(g)^{-1}=s_a$, where a is short, hence $\mathrm{Graphaut}(C)=\{1\}$ and $N(\mathbb{R}) = \mathbb{R}^+ \times W_G$, as asserted.

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