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CONSTRAINED HAMILTONIANS: A HOMOLOGICAL APPROACH

James Stasheff

In recent years, there has been a tremendous revitalization of the historically crucial interaction between mathematics and physics. This is well symbolized by the list of topics at the Winter School at SRNI.

I am very grateful to V. Souček for encouraging me to present this paper for the proceedings of the 1987 Winter School and especially to rework [13] in a more bilingual version so as to increase its accessibility to physicists as well as to mathematicians. The recognition of physical "ghosts" as generators of the Koszul complex of commutative algebra will hopefully prove as fruitful as the recognition of gauge potentials as the connections of differential geometry. "Strong homotopy representations" are introduced as an interpretation of the terms of higher order of Fradkin, Batalin, Vilkovisky and others; these constructs appear to have physical applications beyond the present work.

This revision of [13] has benefitted from conversations with Herbert Neuberger, but any failures of the following exposition to sound colloquial to physicists are entirely mine - physics was not the language the muse sang at my cradle.

The mathematics which is my native tongue is that of algebraic topology and in particular "rational" homomotopy theory which describes the world in terms of the cohomology of differential forms, de Rham cohomology. A significant part of theoretical physics, especially gauge field theory, also describes the world in terms of differential forms, though the vocabulary of cohomology is less prominent. Thus the relevance of cohomology to physics is quite reasonable though unappreciated until recent years. It took some time to recognize the cohomological aspect of Dirac's treatment of the magnetic monopole - discovering, in physical theory, the Hopf fibration

$S^1 \to S^3 \to S^2$ in 1931!

With the development of gauge field theories such as Yang-Mills, de Rham cohomology and characteristic classes became of considerable interest as did other cohomologies, especially those for smooth groups and Lie algebras and representations. Today I'd like to tell you about some developments of the past decade in which the simplicity of Lie algebra representation has given way to a more subtle structure and added homological algebra to the list of physically relevant tools. The generalization which physical intuition forced on those doing certain calculations in gravity turned out to be the same generalization that occurred quite independently in the deformation theory of algebras and rational homotopy theory, in particular, in my work with Mike Schlessinger.

To begin on the physical side, where I speak as a tourist, first there was the work of Fradkin and his school, particularly Batalin and Vilkovisky [6,1,2]. An excellent report by Henneaux [10] called attention to the homological aspect of their technique which was further elucidated by McMullan [5,12] who recognized the Koszul complex and its crucial role therein.

The problem is posed in the setting of the Hamiltonian formalism, which includes the following crucial (for us) ingredients:

A phase space W, e.g., the cotangent bundle T^*M or more generally a symplectic manifold, i.e., a smooth manifold with a closed 2-form of maximal rank ($dp_i \wedge dq_i$ in local coordinates). This gives a Poisson algebra structure on $C^\infty(W)$, i.e., a Poisson bracket $\{\ ,\ \} : C^\infty(W) \otimes C^\infty(W) \longrightarrow C^\infty(W)$ making $C^\infty(W)$ a Lie algebra over R and, in addition, satisfying

$$\{f,gh\} = \{f,g\}h + g\{f,h\} .$$

Thus $\{f,\ \}$ is a derivation of $C^\infty(W)$ and so can be identified with a vector field denoted X_f.

In field theory, W itself can be a space of functions or of sections of a bundle.

The problem that led to the homological algebra considered here is the problem of "constraints", i.e., a family of functions $\{\varphi_\alpha\}$ such that the dynamics is constrained to stay in the submanifold $V \subset W$ which is the common zero set of the φ_α.

Worse yet, the submanifold V does not reflect the true "physical degrees of freedom", but rather the constraints also act on V via their associated vector fields X and this action takes flows (solutions) to physically equivalent ones. The constraints have determined the foliation $\mathcal{F}$ of V and the quotient $V/\mathcal{F}$ is the "reduced phase space" which is the physically meaningful symplectic manifold.

For example:

Gauge theory: Here W is $T^*\mathcal{A}$ where $\mathcal{A}$ is the space of connections for a fixed principal G-bundle $G \to P \to X$. The reduced phase space is $T^*(\mathcal{A}/\mathcal{G})$ where $\mathcal{G}$ is the group of "gauge transformations", i.e., vertical automorphisms of P, that is,

$$\mathcal{G} = \mathrm{Sec}(P x_{Ad} G) .$$

Gravity: Here W is $T^*\mathcal{M}$ where $\mathcal{M}$ is the space of metrics on a fixed manifold X. The reduced phase space is $T^*(\mathcal{M}/\mathrm{Diff}\ X)$.

String theory: Here W is the cotangent bundle to the space of imbeddings or immersions of a Riemannian surface S into a Riemannian manifold M.

Non-linear sigma models: Here W is the cotangent bundle to the space of maps $\mathrm{Map}(M,T)$ of one manifold to another.

Now, for a mathematician, there's a tendency to think that we have a perfectly adequate description of the reduced phase space, but to a physicist, since the original data are on W or even $\mathcal{A}$ and $\mathcal{M}$, there is a need for handling computations in terms of W directly - without passing to the quotient. The method physicists have developed is to add ghost fields to the situation - you can't see ghosts experimentally, but they account computationally for what you do see - and implementing a BRS(T) transformation.

What is a BRS(T) operator? First, BRS refers to Becchi-Rouet- Stora [3], while the T refers to Tyutin, whose preprint has never been published, nor have I been able to locate a copy. At first, the BRS operator appeared to be a formal construct corresponding to a symmetry of certain Lagrangians. Later, it was interpreted in terms of the coboundary operator δ_g

of the Cartain-Chevelley-Eilenberg complex which defines Lie algebra cohomology. We will review this in a moment after explaining why it is relevant to the constraint problem.

We can diagram our situation as

$$\begin{array}{c} V \subset W \\ \downarrow \\ V/\mathcal{F} \end{array} ,$$

where V is the set of common zeroes of the constraints φ_α, i.e., $V = \{v \in W \mid \varphi_\alpha(v) = 0 \text{ , all } \alpha\}$. In terms of $C^\infty(W)$, the algebra of functions on W, the algebra of functions on V can be described as $C^\infty(W)/I$ where I is the ideal generated by the constraints, the ideal of functions which vanish on V, i.e., $I = \{ f^\alpha \varphi_\alpha, f^\alpha \in C^\infty(V)\}$ where the sum is understood to be finite. If $\mathcal{F}$ were given by a group action, $C^\infty(V/\mathcal{F})$ would be given by the G-invariant functions on V, i.e., those $h \in C^\infty(V)$ such that $h(gv) = h(v)$. For a connected Lie group G, it would be sufficient to demand infinitesimal invariance, i.e., invariance under the action of the Lie algebra $\mathfrak{g}$ of G. This is precisely what Lie algebra cohomology computes:

$$\{\mathfrak{g}\text{-invariant } h \in C^\infty(V)\} \approx H^0_{Lie}(\mathfrak{g}, C^\infty(V))$$

$$\approx H^0_{Lie}(\mathfrak{g}, C^\infty(W)/I) .$$

In general, however, the real (or complex) linear span of the constraints do NOT form a Lie algebra under Poisson bracket. Throughout this paper, I will restrict myself to the physically difficult case of FIRST CLASS constraints, meaning $\{\varphi_\alpha, \varphi_\beta\} = C^\gamma_{\alpha\beta}\varphi_\gamma$. If the $C^\gamma_{\alpha\beta}$ were constants, this would describe a Lie algebra, but in general they also are functions on W. One says in physics that the algebra of constraints does not close - this is the trouble we wish to confront.

The ideal I does close under Poisson bracket since $C^\gamma_{\alpha\beta}\varphi_\gamma$ is again in I and

$$(*) \qquad \{\varphi_\alpha, f\varphi_\beta\} = \{\varphi_\alpha, f\}\varphi_\beta + f\{\varphi_\alpha, \varphi_\beta\}$$

is also. Thus $H_{Lie}(I, C^\infty(W)/I)$ can be defined, as we are about to do, and $H^0_{Lie}(I, C^\infty(W)/I)$ will give the desired description of

$C^\infty(V/\mathcal{F})$.

We now recall the formal definition of Cartan-Chevalley-Eilenberg cohomology of a Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module M . This latter means M is a vector space together with a linear map $\theta : \mathfrak{g} \to \mathrm{Aut}\, M$ which is a representation: $\theta[X,Y] = \theta(X)\theta(Y) - \theta(Y)\theta(X)$ for $X,Y \in \mathfrak{g}$.

The Cartan-Chevalley-Eilenberg complex $\mathrm{Alt}(\mathfrak{g},M)$ consists of the <u>alternating multilinear functions from $\mathfrak{g}$ to M</u> , with coboundary $\delta = \delta_g + \delta_\theta$. For an alternating linear function $h : \mathfrak{g}\otimes\ldots\otimes\mathfrak{g} \to M$, define

$$(\delta_g h)(X_o,\ldots,X_p) = \Sigma(-1)^{i+j}h([X_i,X_j],\ldots\hat{i}\ldots\hat{j}\ldots)$$

where $\hat{i}$ denotes omission of X_i and define

$$(\delta_\theta h)(X_o,\ldots,X_p) = \Sigma(-1)^i\theta(X_i)h(\ldots\hat{i}\ldots) .$$

Note that $(\delta_g)^2 = 0$ if and only if $[\ ,\ ]$ satisfies the Jacobi identity, while if $(\delta_g)^2 = 0$, then $(\delta_g+\delta_\theta)^2 = 0$ iff θ is a representation:

$$\theta([X,Y]) = \theta(X)\theta(Y) - \theta(Y)\theta(X) .$$

Given all of this,

$$H_{Lie}(g,M) \quad \text{means} \quad \frac{\mathrm{Ker}\ \delta_g+\delta_\theta}{\mathrm{Im}\ \delta_g+\delta_\theta} .$$

In the setting of first class constraints, we then have defined

$$H_{Lie}(I,C^\infty(W)/I).$$

Henceforth we will simplify notation by denoting the algebra $C^\infty(W)$ by A .

Now instead of considering all alternating multilinear functions of I , Fradkin et al want to use just the complex linear span $\Phi = \{c^\alpha \varphi_\alpha, c^\alpha \in \mathbb{C}\}$ of the constraints. An alternating multilinear function

$$h : \Phi \otimes \ldots \otimes \Phi \to A/I$$

can be extended to $I \otimes \ldots \otimes I$ by requiring A-multilinearity, i.e.,

$$h(x_1, \ldots, x_p) = \sum f_1^{\alpha_1} \ldots f_p^{\alpha_p} h(\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_p})$$

if $x_i = f_i^\alpha \varphi_\alpha$. Thus interpreted, these functions do form a <u>sub-complex</u> of $\mathrm{Alt}(I, A/I)$ (though this is not true for a general I-module) and give the same H°. One of our new insights into the FBV formalisms is the relevance of this subcomplex.

But more is desired, namely, a mechanism for dealing with functions $f \in A$ rather than equivalence classes in A/I (in physical terms, working "off-shell" rather than just "on-shell", V being the shell). To do this, Fradkin et al introduced <u>ghosts</u>.

What are ghosts? They are generators ρ_α of a Grassmann algebra over A in one-to-one correspondence with the constraints φ_α. In this context of FIRST CLASS constraints, I will refer to them as KOSZUL GHOSTS for the ideal I, for indeed the Koszul complex $K(I)$

$$K(I) = A \otimes \Lambda \rho_\alpha$$

is just this Grassmann algebra over A generated by the ρ_α with $d_K : K(I) \longrightarrow K(I)$ being the A-<u>derivation</u> defined by $d_K \rho_\alpha = \varphi_\alpha$.

In more detail, elements of this Grassmann algebra are represented by polynomials

$$f^{\alpha_1 \ldots \alpha_q} \rho_{\alpha_1} \wedge \ldots \wedge \rho_{\alpha_q}$$

where $\alpha_1 < \ldots < \alpha_q$, (the α having been ordered arbitrarily) and $f^{\alpha_1 \ldots \alpha_q} \in A$. Multiplication is determined by $\rho_\alpha \wedge \rho_\beta =$

$-\rho_\beta \wedge \rho_\alpha$. The ghost degree of ρ_α is one, that of the monomial above is q and d_K is a graded A-derivation with respect to ghost degree so that d_K of the above monomial is

$$\sum_i (-1)^{i-1} \varphi_{\alpha_i} f^{\alpha_1 \ldots \alpha_q} \rho_{\alpha_1} \wedge \ldots \hat{\rho}_{\alpha_i} \ldots \wedge \rho_{\alpha_q} .$$

A word about notation: In the physics literature, with the exception of Browning and McMullan, the derivation d_K is written $\varphi_\alpha \eta^\alpha$ where η^α is interpreted as dual to ρ_α , i.e., $\langle \eta^\alpha, \rho_\beta \rangle = \delta^\alpha_\beta$. The Koszul complex appears as $A \otimes \Lambda\rho_\alpha$.

Fradkin et al, Henneaux and Browning and McMullan also consider the possibility of starting with fermions as well as bosons, i.e., starting with a graded commutative algebra A with homogeneous constraints φ_α . The ghost ρ_α then has the opposite parity to φ_α , in fact ghost degree ρ_α = degree $\varphi_\alpha + 1$.

The point of the Koszul complex (as introduced by Koszul) is that if I is a regular ideal (in our situation, this is implied by 0 being a regular value of $W \xrightarrow[\varphi]{} \mathbb{R}^N$ where the components of φ are the constraints φ_α), then the homology of $K(I)$ is A/I in ghost degree zero and is 0 otherwise (d_K-closed => d_K-exact for ghost degree > 0 and in ghost degree 0 , the image of d_K is precisely I.)

There is in fact a contracting homotopy $s_K : K(I) \to K(I)$ of ghost degree -1 so that

$$d_K s_K + s_K d_K = 1_K - \bar{\pi}$$

where 1_K is the identity on $K(I)$ and $\bar{\pi} : K(I) \xrightarrow{\bar{\pi}} A/I \to A \subset K(I)$ sends all ρ_α to zero, sends $A \to A/I$ by the quotient map and then maps $A/I \to A$ as a complement to I . Formulas for s_K can be given in a quite explicit but complicated way. Having introduced $K(I)$, we can now substitute it for A/I and consider

$$\mathrm{Alt}(\Phi, K(I))$$

the alternating multilinear functions from Φ into $K(I)$ or, to stay closer to the FBV notation,

$$\Lambda\eta^{\alpha} \otimes A \otimes \Lambda\rho_{\alpha} .$$

Here the η_{α} are called <u>anti-ghosts</u>, have degree -1 and, for our purposes, need be interpreted only as the linear duals of the φ_{α}. Thus an alternating function $h : \Phi \otimes \ldots \otimes \Phi \to K(I)$ of p-variables and of ghost degree q can be denoted

$$\eta^{\alpha_1 \ldots \alpha_p} \; f^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \; \rho_{\beta_1} \ldots \rho_{\beta_q} .$$

(When there are infinitely many ghosts, h may have infinitely many such terms as the notation Alt helps remind us.)

Now to define a Cartan-Chevalley-Eilenberg differential. Here, we have two problems: Φ is NOT closed under $\{\ ,\ \}$ (in physics it is called an open algebra) unless the structure functions $C^{\gamma}_{\alpha\beta}$ are constants. We can still define δ_{Φ} using A-linearity; in other words, for $h : \Phi \to A$, for example,

$$(\delta_{\Phi} h)(\varphi_{\alpha},\varphi_{\beta}) = - C^{\gamma}_{\alpha\beta} h(\varphi_{\gamma}) .$$

However, $\delta^2_{\Phi} \neq 0$. Even worse, the obvious representation of I on $A \subset K(I)$ will not do because it does not commute with d_K:

$$\{\varphi_{\beta},f\} \otimes \rho_{\alpha} = \{\varphi_{\beta},f\} \otimes \rho_{\alpha}$$

$$d_K\{\varphi_{\beta},f \otimes \rho_{\alpha}\} = \{\varphi_{\beta},f\}\varphi_{\alpha}$$

but $\{\varphi_{\beta},d_K(f \otimes \rho_{\alpha})\} = \{\varphi_{\beta},f\varphi_{\alpha}\} = \{\varphi_{\beta},f\}\varphi_{\alpha} + f\{\varphi_{\beta},\varphi_{\alpha}\}$. This suggests we define $\theta(\varphi_{\alpha})$ by

$$\theta(\varphi_{\alpha})\{f \otimes \rho_{\beta}\} = \{\varphi_{\alpha},f\} \otimes \rho_{\beta} + fC^{\gamma}_{\alpha\beta} \otimes \rho_{\gamma}$$

and extend $\theta(\varphi_{\alpha})$ to all of $K(I) = A \otimes \Lambda\rho_{\alpha}$ as a graded derivation. However $\theta : \Phi \to \text{Aut } K(I)$ is not a representation and even $d_K + \delta_{\Phi} + \delta_{\theta}$ does not have square zero. What to do?

Fradkin, Batalin, Vilkovisky and Henneaux get around these problems by adding terms of higher order. In their notation:

$$d_K \longleftrightarrow \varphi_{\alpha}\eta^{\alpha}$$

$$d_\Phi + d_\theta \longleftrightarrow -\frac{1}{2} C^\gamma_{\alpha\beta}\, \eta^\alpha \eta^\beta \rho_\gamma + \{\varphi_\alpha, \cdot\}\eta^\alpha ,$$

since $d_K(\rho_\beta) = \varphi_\beta = \varphi_\alpha \delta^\alpha_\beta = \varphi_\alpha \eta^\alpha(\rho_\beta)$, etc. They and Henneaux prove:

Theorem. There exist polynomials

$$U^{\alpha_1 \ldots \alpha_i} \quad \text{in} \quad \Lambda\eta^\alpha \otimes A \quad \text{such that}$$

$U^{\alpha_1 \ldots \alpha_i} \rho_{\alpha_1} \ldots \rho_{\alpha_i}$ has ghost degree plus anti-ghost degree equal to -1 and

$$\Omega = \varphi_\alpha \eta^\alpha + \{\varphi_\alpha, \ \}\eta^\alpha + \sum_I U^{\alpha_I} \rho_{\alpha_I}, \ I = \alpha_1 \quad \alpha_2 \ldots < \alpha_i$$

satisfies

$$\{\Omega, \Omega\} = 0 .$$

Browning and McMullan have translated this into differential notation:

Theorem: There exist derivations δ_i which increase the number of ghosts by $i-1$ and the number of anti-ghosts by i such that $D^2 = 0$ for $D = d_K + \delta_\Phi + \delta_\theta + \delta_2 + \delta_3 + \ldots$.

The precise correspondence is given as follows (compare Browning and McMullan): Since $\Lambda\eta_\alpha \otimes A \otimes \Lambda\rho^\alpha$ is freely generated over A by η_α and ρ_α , it is enough to describe $\delta_i \eta^\alpha$, $\delta_i \rho_\alpha$ and $\delta_i f$ for $f \in A$:

$$d_K \rho_\alpha = \varphi_\alpha$$

$$\delta_\Phi \eta^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta}\, \eta^\alpha \eta^\beta$$

$$\delta_\theta \rho_\alpha = C^\gamma_{\alpha\beta}\, \eta^\beta \rho_\gamma$$

$$\delta_\theta f = \{\gamma_\alpha, f\}\eta^\alpha .$$

Similarly

$$\delta_i f = \eta^{\underline{\beta}}\{U^{\bar{\alpha}}_{\underline{\beta}}, f\}\rho_{\bar{\alpha}}$$

$$\delta_i \eta^\alpha = (-1)^{i+1}\, i\, \eta^{\underline{\beta}}\, U^{\alpha\bar{\alpha}}_{\underline{\beta}}\, \rho_\alpha$$

$$\delta_i \rho_\beta = (i+1)\, \eta^{\underline{\beta}}\, U^{\bar{\alpha}}_{\beta\underline{\beta}}\, \rho_{\bar{\alpha}}$$

where $\bar{\alpha} = \alpha_1 \ldots \alpha_j$ $\quad j = i-1,\ i-1$ or i , respectively, $\underline{\beta} = \beta_1 \ldots \beta_k$ $\quad k = i,\ i+1$ or $i+1$.

(The signs and integer factors are an artifact of the Grassmann algebra.)

Fradkin et al find these terms of higher order in succession by setting up and solving a system of ODE's. Henneaux makes use of the resolution property of $K(I)$: each U^{α_I} exists essentially because the obstruction to its existence can be computed to be a cycle which in $K(I)$ is automatically a boundary. The computations are quite complex and, at each stage, choices must be made. Looking at Browning and McMullan's exposition inspired me to apply the techniques of <u>homological perturbation theory</u>, and thus to see that only one choice is necessary (a contracting homotopy for $K(I)$) and the existence of terms of higher order is guaranteed by the general results of homological perturbation theory.

The word perturbation is due to one of the originators, V. K. A. M. Gugenheim [7], inspired by the term as used in physics, but not in his wildest dreams did he believe his theory would apply to physics!

Homological perturbation theory has developed gradually in a series of papers. Essential points in its development are in the papers of Gugenheim [7] and in his papers with May [8] and Stasheff [9]. Hopefully, Gugenheim, Lambe and Stasheff will soon have a summary and exposition of the fully developed theory.

In essence, the theory involves strong deformation retraction (SDR) data for comparison of two complexes with additional structure. In our case, $\Lambda\eta^\alpha \otimes A/I$ with zero differential and $\Lambda\eta^\alpha \otimes A \otimes \Lambda\rho_\alpha$ with respect to d_K have the same homology and we wish to find δ_i as in the Theorem so that $D = d_K + \delta_\Phi + \delta_\theta + \delta_1 + \ldots$ will calculate the $\delta_I + \delta_\theta$ cohomology of $\Lambda\eta^\alpha \otimes A/I$.

Remember that, although Φ is an open algebra, it closes in its representation on A/I (the deviation from closure in A is at least in I). The problem therefore is really one of lifting the representation $\Phi \to \mathrm{Aut}(A/I)$ to $\mathrm{Aut}(K(I)$ or, if not to a representation, to something which will suffice homologically.

Here the general philosophy of homotopy theory is relevant: recall that we lifted θ to $\theta : \Phi \to K(I)$ as a derivation defined by

$$\theta(\varphi_\alpha)\{f \otimes \rho_\beta\} = \{\varphi_\alpha, f\} \otimes \rho_\beta + fC^\gamma_{\alpha\beta} \otimes \rho_\gamma$$

but that θ is not a representation: $\theta(\{\varphi_\alpha, \varphi_\beta\})$ is not even defined and, if we do define it by

1) $\theta(\{\varphi_\alpha, \varphi_\beta\}) = C^\gamma_{\alpha\beta}\ \theta(\varphi_\gamma)$

this is not the same as

2) $\theta(\varphi_\alpha)\theta(\varphi_\beta) - \theta(\varphi_\beta)\theta(\varphi_\alpha)$.

However 1) and 2) are both derivations of $K(I)$ and the difference anti-commutes with d_K. Thus we can use s_K applied to the difference to define $\theta^2(\varphi_\alpha, \varphi_\beta)$ of ghost degree 1 so that

$$d_K \circ \theta^2(\varphi_\alpha, \varphi_\beta) + \theta^2(\varphi_\alpha, \varphi_\beta) \circ d_K$$

$$= \theta(\{\varphi_\alpha, \varphi_\beta\}) - \theta(\varphi_\alpha)\theta(\varphi_\beta) + \theta(\varphi_\beta)\theta(\varphi_\alpha) .$$

Now θ^2 can be added to form $d_K + \delta_\Phi + \delta_\theta + \delta_2$ which may still not have square zero, but is at least zero to one higher order, meaning ghost degree. One is tempted to iterate.

This is what homological perturbation theory is all about. The method is inductive, indeed algorithmic, involving only one choice - that of the contracting homotopy s_K. The result is what is known as a "strong-homotopy-representation" (shr), i.e., a family $\{\theta^i\}$ of alternating maps

$$\theta^i : \Phi \otimes \ldots \otimes \Phi \longrightarrow \mathrm{Der}(K(I))$$

such that

$$\theta^i(\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_i})$$

is a graded derivation of ghost degree $i-1$, that is, θ^i increases ghost degree by $i-1$ and
$\theta^i(\ldots)(X\wedge Y) = \theta^i(\ldots)(X)\wedge Y + (-1)^{(i-1)\deg X} X\wedge\theta^i(\ldots)Y$.

The total differential $D = d_K + \delta_\Phi + \delta_\theta + \delta_2 + \ldots$ can then be expressed more explicitly for $h \in \mathrm{Alt}^p(\Phi, K(I))$ by

$$(*) \qquad (\delta_i h)(\varphi_{\alpha_0}, \ldots, \varphi_{\alpha_{(p+i)}}) = \sum (-1)^{\sigma(\alpha_I)} \theta^i(\varphi_{\alpha_{j_1}}, \ldots, \varphi_{\alpha_{j_i}})\, h(\ldots\hat{\varphi}_{\alpha_{j_1}} \ldots \hat{\varphi}_{\alpha_{j_i}} \ldots)$$

where $\alpha_I = (\alpha_{j_1} < \ldots < \alpha_{j_i})$ and $\sigma(\alpha_I)$ denotes the permutation which puts $\alpha_{j_1} < \ldots < \alpha_{j_i}$ in order in front of the remaining indices.

Since δ_i is a derivation, it is enough to consider $h = f$ or η^α or ρ_β ; the formulas of Browning and McMullan given above then express (*).

But what of the physical interpretation! First, does this choice imply the theory is ambiguous? No, if we regard $\Lambda\eta^\alpha \otimes A \otimes \Lambda\rho_\alpha$ as an extended phase space, then the choices correspond precisely to a "canonical" change of coordinates: $(p,q) \longrightarrow (p',q')$. Here the original $(p,q) \in W$ are augmented by regarding $(\rho_\alpha, \eta^\alpha)$ as a canonical pair - momentum and position, respectively. Indeed, the commutation rules are given by the duality if we set
$\{\eta^\beta, \rho_\alpha\} = \eta^\beta(\rho_\alpha) = \delta^\beta_\alpha$.

So far, I've talked entirely in terms of constraints. The Hamiltonian formalism refers to a dynamics given by the differential equation

$$\frac{d\varphi}{dt} = \{H,\varphi\} .$$

A physical problem begins with $H \in A = C^{\infty}(W)$ such that $\{H,\varphi_{\alpha}\} \in I$ for all constraints φ_{α}. To treat $\Lambda\eta^{\alpha} \otimes A \otimes \Lambda\rho_{\alpha}$ as an extended phase space, we need to extend H to $\bar{H}$ of total ghost degree zero with $D\bar{H} = 0$. Again, this is precisely what is guaranteed by homological perturbation theory, or more simply, by its end result that $H(\Lambda\eta^{\alpha} \otimes A/I))$ is the same as the D-homology of $\Lambda\eta^{\alpha} \otimes A \otimes \Lambda\rho_{\alpha}$.

What about quantization? Everything seems to start smoothly. Observables are identified with elements of the homology in total ghost degree zero (i.e., the number of ghosts ρ = the number of anti-ghosts η). States are more of a problem. They should be (equivalence classes of) functions of the η^{α} but not the ρ_{α}. The ghosts ρ_{α} and antighosts η^{α} are then implemented as operators on (representatives of) states:

$$\hat{\eta}^{\alpha} = \text{multiplication by } \eta^{\alpha} ,$$

$$\hat{\rho}_{\alpha} = -i\partial/\partial\eta^{\alpha} .$$

Physical states should be represented by $\hat{D}$ cycles: $\hat{D}|\varphi\rangle = 0$. (The operator $\hat{D}$ is obtained from $D = \varphi_{\alpha}\eta^{\alpha} + U^{\alpha}{}_{I}\rho_{I}$ by $\eta \to \hat{\eta}$, $\rho \to \hat{\rho}$.) Two representatives φ_1,φ_2 are physically equivalent if $\varphi_1 - \varphi_2 = \hat{D}\chi$. There remain significant subtleties with regard to the approprite inner product and/or completion.

As elsewhere in contemporary physics, it is time to treat seriously the category of differential graded Hilbert spaces.

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