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Riemann-Roch theorem and Lie algebra cohomology I

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Introduction

All associative algebras and Lie algebras in this paper are defined over the complex field $\mathbb{C}$.

Let L be the Lie algebra of vector fields on the circle. An element of L is a field $f(\varphi)\frac{d}{d\varphi}$ where $f(\varphi)$ is a Fourier polynomial. Denote the module of tensor fields of type λ by F_λ, $\lambda \in \mathbb{C}$. An element of F_λ is an expression $g(\varphi)(d/d\varphi)^\lambda$, and $(f\cdot d/d\varphi)\cdot(g\cdot (d/d\varphi)^\lambda) = (fg' - \lambda f'g)(d/d\varphi)^\lambda$. Here g is also a Fourier polynomial. Fix the decomposition $F_\lambda = V_+ \oplus V_-$ where $V_+ = \{g(\varphi)(d/d\varphi)^\lambda : g(\varphi) = \sum_{s \geq 0} a_s e^{(2\pi i)s\varphi}\}$ and $V_- = \{g(\varphi)(d/d\varphi)^\lambda : g(\varphi) = \sum_{s<0} a_s e^{(2\pi i)s\varphi}\}$. Let P be the projection operator $V \to V_-$ along V_+. Define a map $\Theta : L \to \operatorname{End} V_-$ as follows. Put $\Theta(X)Y =$

$= P(X(Y))$, $X \in L$, $Y \in V_-$; $X(Y)$ is the result of the action of X on the tensor field Y. The map θ is "almost a representation", i.e., $\mathrm{Im}(\theta([X, Y]) - [\theta(X), \theta(Y)])$ is finite dimensional. Put $w(X, Y) = \mathrm{tr}(\theta([X, Y]) - [\theta(X), \theta(Y)])$. It is well known that w is a cocycle representing the cohomology class $-2\cdot(6\lambda^2 + 6\lambda + 1)\cdot c$ where c is generator of $H^2(L)$ given by the form (cf. [GF])

$$C\left(f\frac{d}{d\varphi}, g\frac{d}{d\varphi}\right) = \frac{1}{2\pi i}\int (f'g'' - f''g')\frac{d}{12}. \qquad (1)$$

This statement has an equivalent form. Let $\hat{L}$ be the Virasoro algebra which is the central extension of L corresponding to c. There is the natural pairing $F_\lambda \times F_{-1-\lambda} \xrightarrow{\varkappa} \mathbb{C}$; $(g_1(d/d\varphi)^\lambda, g_2(d/d\varphi)^{-1-\lambda}) \to \int g_1 g_2 d\varphi$. Let $\overline{V}_+$, $\overline{V}_-$ be the annihilators of V_+, V_- respectively. The pairing $\varkappa$ determines the quadratic form on $F_\lambda + F_{-1-\lambda}$: $\langle u+v, u+v\rangle = \varkappa(u, v)$. Put $W_- = V_- + \overline{V}_-$ and let H be the representation of the Clifford algebra associated to the form $\varkappa$ such that there is a vector $v \in H$, $W_-\cdot v = 0$. As it is well known ([FF]), there is the action of $\hat{L}$ on H uniquely determined by the following conditions:

a) $\hat{L}$ is contained in the normalizer on W.

b) W is isomorphic to $F_\lambda + F_{-1-\lambda}$ as an L-module. The central charge (i.e., the action of the central element of $\hat{L}$ on H) is equal to $-2\cdot(6\lambda^2 + 6\lambda + 1)$.

The polynomial $-2(6\lambda^2 + 6\lambda + 1)$ appears frequently in [ADKP], [BS], [P] as the gravitational anomaly in two-dimensional conformal field theory or in representation theory of Virasoro algebra. It is also closely related to Riemann-Roch theorem. Namely, let $X \xrightarrow{\pi} S$ be a family of Riemann surfaces. Then

$$c_1(\pi_! \mathcal{T}^\lambda_{X/S}) = (6\lambda^2 + 6\lambda + 1)\, c_1(\pi_! \mathcal{O}) \qquad (2)$$

where $\pi_!$ is the direct image in K-theory and $\mathcal{T}_{X/S}$ is the relative tangent bundle.

All these results are related to the problem of finding "local" proof of Riemann-Roch theorem or index theorem. The examples of such considerations may be found in [BS], [ADKP] where the Riemann-Roch-Grothendieck theorem for one-dimensional families is deduced from the purely local facts on Lie algebra cohomology of vector fields. Our aim is to obtain corresponding local statement for arbitrary families.

Let $\mathrm{Diff}(S^1)$ be the algebra of differential operators on the circle whose coefficients are Fourier polynomials. Let $\mathfrak{gl}(\mathrm{Diff}(S^1))$ be the Lie algebra of finite matrices over $\mathrm{Diff}(S^1)$. As it may be deduced from the results of [BG], [FT], the cohomology $H^*(\mathrm{Diff}(S^1))$ is the free skew commutative graded algebra with generators in dimensions 2, 3, 4,... . Denote by η_α the generator in dimension α.

It has been shown in [GF1] that $H^*(L)$ is freely generated by c, ν where $\deg c = 2$ and $\deg \nu = 3$. The action of L on F determines the embedding $\varphi_\lambda : L \to \mathfrak{gl}_1(\mathrm{Diff}(S^1)) \hookrightarrow \mathfrak{gl}(\mathrm{Diff}(S^1))$. One has

$$\varphi_\lambda^*(\eta_2) = -2(6\lambda^2 + 6\lambda + 1)\cdot Kc \tag{3}$$

$$\varphi_\lambda^*(\eta_3) = -2(6\lambda^2 + 6\lambda + 1)\cdot K'\nu \tag{4}$$

where K, K' does not depend on λ.

The algebra $\mathrm{Diff}(S^1)$ contains the subalgebra isomorphic to algebra Diff_1 of differential operators on $\mathbb{C}$ with polynomial coefficients. This subalgebra comprises the operators whose coefficients are of the form $\sum_{s \geq 0} a_s e^{(2\pi i)s}$. The intersection of Diff_1 and L is isomorphic to the Lie algebra W_1 of vector fields on $\mathbb{C}$ with polynomial coefficients. According to [FT1] the cohomology of $\mathfrak{gl}(\mathrm{Diff}_1)$ is the free skew commutative graded algebra generated by $\mathcal{E}_\alpha$, $\alpha = 3, 5, 7, \ldots$, $\deg \mathcal{E}_\alpha = \alpha$. Consider the diagram of embeddings:

$$\begin{tikzcd} W_1 \arrow[r, "\varphi_\lambda"] \arrow[d] & \mathrm{Diff}_1 \arrow[r] \arrow[d] & \mathfrak{gl}(\mathrm{Diff}_1) \arrow[d] \\ L \arrow[r, "\varphi_\lambda"] & \mathrm{Diff}(S^1) \arrow[r] & \mathfrak{gl}(\mathrm{Diff}(S^1)) \end{tikzcd}$$

(Here a map $\mathrm{Diff} \longrightarrow \mathfrak{gl}(\mathrm{Diff})$ acts as follows: $X \longrightarrow X \cdot E_{11}$, where E_{11} is a matrix entry. For any odd α, the restriction of r_α to $\mathfrak{gl}(\mathrm{Diff}_1)$ is ξ_α. The cohomology of W_1 is nonzero only in dimensions 0 and 3; the map $H^3(L) \longrightarrow H^3(W_1)$ is an isomorphism ([GF]). Thus, instead of studying the embedding $L \longrightarrow \mathfrak{gl}(\mathrm{Diff}(S^1))$ we may consider purely local embedding $W_1 \rightarrow \mathfrak{gl}(\mathrm{Diff}_1)$. For any Lie algebra L, there is a homomorphism $H^i(L) \rightarrow H^{i-1}(L, L^*)$. Consider the commutative diagram

$$\begin{tikzcd} H^3(\mathfrak{gl}(\mathrm{Diff}_1)) \arrow[r, "\varphi_\lambda^*"] \arrow[d] & H^3(W_1) \arrow[d] \\ H^2(\mathfrak{gl}(\mathrm{Diff}_1), \mathfrak{gl}(\mathrm{Diff}_1)^*) \arrow[r, "\varphi_\lambda^*"] & H^2(W_1, W_1^*) \end{tikzcd}$$

It follows from [FT1] and [F] that all arrows here are isomorphisms. Thus, formula (4) is equivalent to the following: if α, β are generators of $H^2(W_1, W_1^*)$ and $H^2(\mathfrak{gl}(\mathrm{Diff}_1), \mathfrak{gl}(\mathrm{Diff}_1)^*)$ respectively, then

$$\varphi_\lambda^*(\beta) = -2(6\lambda^2 + 6\lambda + 1) \cdot K' \tag{5}$$

where K' does not depend on λ. It is not hard to show that (3) is also a consequence of (5).

The statement about the coefficient $-2(6\lambda^2 + 6\lambda + 1)$ may be generalized to higher dimensions as follows. Let Diff_n be the algebra of differential operators with polynomial coefficients and W_n be the Lie algebra of vector fields on $\mathbb{C}^n$ with polynomial coefficients. Let λ be a finite dimensional representation of $\mathfrak{gl}_n$. Denote by F_λ the space of tensor fields of type λ. The action of

W_n on F_λ provides the embedding $\varphi_\lambda : W_n \to \mathfrak{gl}_{\dim_\lambda}(\mathrm{Diff}_n) \to \mathfrak{gl}(\mathrm{Diff}_n)$. Throughout the paper, we denote $\mathfrak{gl}(\mathrm{Diff}_n)$ by D_n. Consider the commutative diagram

$$\begin{array}{ccc} H^{2n+1}(D_n) & \xrightarrow{\varphi_\lambda^*} & H^{2n+1}(W_n) \\ \downarrow & & \downarrow \\ H^{2n}(D_n, D_n^*) & \xrightarrow{\varphi_\lambda^*} & H^{2n}(W_n, W_n^*) \end{array}$$

It has been shown in [F], [FT1] that the vertical arrows are bijective and $H^{2n+1}(D_n)$ is one-dimensional.

Now recall the basic facts on Gelfand-Fuchs cohomology. Let $p : E \to B_n$ be the universal bundle for the group $GL_n(\mathbb{C})$. Denote by Y_n the 2n-skeleton of B_n; $X_n = p^{-1}B_n$. Then $H^*(W_n) \xrightarrow{\sim} H^*(X_n)$ ([GF]). Consider the boundary map in the exact sequence of the pair (E, X_n): $H^{2n+1}(X_n) \to H^{2n+2}(E/X_n)$. Clearly it is an isomorphism. The map of pairs $(E, X_n) \to (B_n, Y_n)$ induces the homomorphism $H^{2n+2}(B_n/Y_n) \to H^{2n+2}(E/X_n)$ which is also an isomorphism. We obtain that $H^{2n+1}(W_n) \xrightarrow{\sim} H^{2n+2}(B_n/Y_n)$. But the latter space is in turn isomorphic to $H^{2n+2}(B_n)$, i.e., to the space of symmetric polynomials in n variables of degree $n+1$. The representation λ determines the bundle $\mathcal{T}^\lambda$ on B_n. Let $\mathcal{T}$ be the bundle corresponding to the standard n-dimensional representation of $\mathfrak{gl}_n$. Now, the "local Riemann-Roch theorem" in this partial case states that the image of the generator of $H^{2n+1}(D_n)$ under the composition

$$H^{2n+1}(\mathfrak{gl}(\mathrm{Diff}_n)) \to H^{2n+1}(W_n) \xrightarrow{\sim} H^{2n+2}(B_n) \tag{6}$$

is equal to $(\mathrm{ch}\,\mathcal{T}^\lambda \cdot \mathrm{td}\,\mathcal{T})_{n+1}$ where ch is the Chern character, td is the Todd genus and the subscript $n+1$ means that we take the component in H^{2n+2}. The particular case of Riemann-Roch-Grothendieck theorem stating that

$$c_1(\pi_! \mathcal{T}^\lambda_{X/S}) = \pi_* (\mathrm{ch}\, \mathcal{T}^\lambda_{X/S} \cdot \mathrm{td}\, \mathcal{T}_{X/S}) \tag{7}$$

may be deduced from the previous result. We hope to discuss this elsewhere.

We may obtain an equivalent statement passing to relative Lie algebra cohomology. Consider the subalgebra $\mathfrak{gl}_n \subset W_n$ comprising the fields $\sum a_{ij} x_i\, d/dx_j$, $a_{ij} \in \mathbb{C}$. It is easy to see that $H^{2n}(W_n, W_n^*) \longrightarrow H^{2n}(W_n, \mathfrak{gl}_n; W_n^*)$ and $H^{2n}(D_n, \mathfrak{gl}_n; D_n^*) \simeq \mathbb{C}$. Thus, the image of 1 under the composition

$$\mathbb{C} \to H^{2n}(D_n, \mathfrak{gl}_n; D_n^*) \longrightarrow H^{2n}(W_n, \mathfrak{gl}_n; W_n^*) \xrightarrow{\sim} H^{2n+2}(B_n) \tag{8}$$

is equal to $(\mathrm{ch}\, \mathcal{T}^\lambda \cdot \mathrm{td}\, \mathcal{T})_{n+1}$. This form of the "local Riemann-Roch theorem about $c_1(\pi_! \mathcal{T})$" is most suitable for generalizing to higher dimensions.

Recall that if ρ is a finite dimensional representation of a Lie algebra $\mathfrak{g}$, i.e., a homomorphism $\mathfrak{g} \to \mathfrak{gl}(\mathbb{C})$, one may define the Chern character of ρ:

$$\mathrm{ch}(\rho) \in S^{**}(\mathfrak{g}^*)^{\mathfrak{g}}; \quad (\mathrm{ch}(\rho))(x) = \mathrm{tr}\, \exp \rho(x).$$

(Here and below we denote $S^{**} = \prod_{j \geq 0} S^j$, etc.) It happens that this construction may be generalized to the representations over the rings A, i.e., to the homomorphisms $L \to \mathfrak{gl}(A)$ when $\mathfrak{g}$ is reductive and A satisfies certain homological condition. Assume that the Hochschild homology $HH_*(A)$ (cf. 1.1) is concentrated in unique dimension 2n, and $HH_{2n}(A) \xrightarrow{\sim} \mathbb{C}$. When $A = \mathbb{C}$ then $n = 0$. We show (Proposition 3.1.2)

$$\begin{aligned} &H^{2n}(\mathfrak{gl}(A), \rho(\mathfrak{g}); S^q \mathfrak{gl}(A)^*) \xrightarrow{\sim} \mathbb{C}, && q > 0; \\ &H^{i}(\mathfrak{gl}(A), \rho(\mathfrak{g}); S^q(\mathfrak{gl}(A)^*) = 0, && q > 0,\ i < 2n. \end{aligned} \tag{9}$$

Consider the relative Weyl algebra $W^*(\mathfrak{gl}(A); \rho(\mathfrak{g}))$ (cf. 1.1). The above statement provides the maps

$$\mathbb{C} \to H^{2n}(\mathfrak{gl}(A), \rho(\mathfrak{g}); S^q \mathfrak{gl}(A)^*) - H^{2(n+q)}(W^*(\mathfrak{gl}(A); \rho(\mathfrak{g})) \qquad (10)$$

On the other hand, one has an isomorphism

$$H^{2i}(W^*(\mathfrak{gl}(A), \rho(\mathfrak{g}))) \to S^i(\rho(\mathfrak{g})^*)^{\rho(\mathfrak{g})}, \ \forall i$$

and thus a homomorphism

$$H^2 \ (W^*(\mathfrak{gl}(A), \rho(\mathfrak{g}))) \to S \ (\mathfrak{g}^*)^{\mathfrak{g}}.$$

Combining this with (10) one obtains the maps

$$\varphi_{n+q} : \mathbb{C} \to S^{n+q}(\mathfrak{g}^*)^{\mathfrak{g}}, \quad q > 0.$$

A simple trick allows to define also φ_j for $j \leq n$. Put

$$\chi(\rho) = \sum_{j=0} \frac{(-1)^j \ \varphi_j(\rho)(1)}{j!}$$

Within our approach, the local Riemann-Roch theorem is the character formula for the special representation of the Lie algebra $\mathfrak{gl}_n \oplus \mathfrak{gl}$ over the associative algebra Diff_n. Namely, let $\mathfrak{gl}_n \in D_n$ as above and $\mathfrak{gl} = \mathfrak{gl}(\mathbb{C}) \hookrightarrow \mathfrak{gl}(\mathrm{Diff}_n) = D_n$; we obtain the Lie algebra homomorphism $\mathfrak{gl}_n \oplus \mathfrak{gl} \xrightarrow{\rho} D_n$. In 3.2 we recall from [FT1] that $HH_{2n}(\mathrm{Diff}_n) \xrightarrow{\sim} \mathbb{C}$ and $HH_i(\mathbb{C}) = 0$, $i \neq 2n$. Thus, we are able to construct $\chi(\rho)$. Identify $S^*(\mathfrak{gl}_n \oplus \mathfrak{gl})^{\mathfrak{gl}_n \oplus \mathfrak{gl}}$ with $H^*(BGL_n \times BGL)$. Put $\mathcal{T} = \tau_n \boxtimes 1$, $\mathcal{E} = 1 \boxtimes \tau$ where τ_n, τ are the universal bundles. The main Theorem 4.1.2 claims that

$$\chi(\rho) = \mathrm{ch}\, \mathcal{E} \cdot \mathrm{td}\, \mathcal{T} \qquad (11)$$

Note that this formulation does not involve the Lie algebra W_n but only D_n.

The local Riemann-Roch theorem for tensor fields is the character formula for the representation ρ which is a composition $\mathfrak{gl}_n \to W_n \xrightarrow{\varphi_\lambda} D_n$. Identify $S^*(\mathfrak{gl}_n)^{\mathfrak{gl}_n}$ with $H^*(BGL_n)$ (or B_n above). Let $\mathcal{T}$, $\mathcal{T}^\lambda$ be as above. Then

$$\mathrm{ch}(\rho_\lambda) = \mathrm{ch}\,\mathcal{T}^\lambda \cdot \mathrm{td}\,\mathcal{T}.$$

The contents of the paper are the following. In §1 we, proceeding in spirit of [ADKP], [F], give a geometric construction which relates the usual Riemann-Roch-Grothendieck theorem to the above local theorem. In §2 we construct the generalized characters of representations. In §3 we make the technical computations concerning the cohomology of Diff_n and D_n. In particular, we select the distinguished generators in $H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; S^q D_n^*)$. In §4 we state and prove the local Riemann-Roch theorem (11). In §5 we study in more detail its particular case - the local Riemann-Roch-Hierzebruch formula. Recall that for any pair $\mathfrak{g} \subset L$ where L is a Lie algebra and $\mathfrak{g}$ a subalgebra reductive in L (cf. 1.1) one may define the Chern-Weyl homomorphism $S(\mathfrak{g}^*) \xrightarrow{c} H^{2*}(L, \mathfrak{g}; \mathbb{C})$ (cf. 5.1). Define the "local Euler characteristic" χ to be the image of the distinguished generator of $H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; D_n^*) \to \mathbb{C}$ under the map $H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; D_n^*) \longrightarrow H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; \mathbb{C})$. Then (Theorem 5.1.1)

$$\chi = c(\mathrm{ch}\,\mathcal{E} \cdot \mathrm{td}\,\mathcal{T})_n.$$

In the beginning of our work we were inspired by the article of Losik [L]. His paper contains a calculation in Weil algebra of Lie algebra of a formal vector fields similar to our.

The first author had lectures in Srni during a winter school "Geometry and physics" about Riemann-Roch and Lie algebra cohomology (January, 1988). I (B.L.F) am grateful to organizers of this school for their hospitality and participants for their interest.

§1. Geometric formulation of the main theorem

1.1. Preliminaries. Here we recall the well known results and constructions from homological algebra.

Let L be a Lie algebra and M be a module over L. Consider the standard complexes

$$C_*(L, M) = \Lambda^*(L)\otimes M; \quad d : C_*(L, M) \longrightarrow C_{*-1}(L, M);$$

$$d(X_1\wedge \dots \wedge X_k \otimes m) = \sum_{1\le i<j\le k} (-1)^{i+j} [X_i, X_j]\wedge \dots \wedge \hat{X}_i \wedge \dots \wedge \hat{X}_j \wedge \dots +$$

$$+ \sum_{1\le i\le k} (-1)^i X_1\wedge \dots \wedge \hat{X}_i \wedge \dots \quad X_k \otimes X_i m; \tag{1}$$

$$C^*(L, M) = \mathrm{Hom}_{\mathbb{C}}(\Lambda^*(L), M); \quad d : C^*(L, M) \to C^{*+1}(L, M);$$

$$(d\omega)(X_1,\dots, X_{k+1}) = \sum_{1\le i<j\le k+1} (-1)^{i+j} \omega([X_i, X_j],\dots, \hat{X}_i,\dots,\hat{X}_j,\dots)+$$

$$+ \sum_{1\le i\le k+1} (-1)^{i-1} X_i\, \omega(X_1,\dots, \hat{X}_i,\dots, X_{k+1}) \tag{2}$$

Put $H_*(L, M) = H_*(C_*(L, M))$; $H^*(L, M) = H^*(C^*(L, M))$ (cf. [CE]). These groups are called the Lie algebra (co)homology groups of L with coeffients in M. Now, let $\mathfrak{g}$ be a Lie subalgebra of L. Assume that $\mathfrak{g}$ is reductive in L, i.e., that $\mathfrak{g}$ is a reductive Lie algebra and L is a direct sum of finite dimensional L-modules with respect to the adjoint action. In this case we define the relative (co)homology $H_*(L,\mathfrak{g}; M)$ to be the (co)homology of the complexes:

$$C_*(L, \mathfrak{g}; M) = (\Lambda^*(L/\mathfrak{g})\otimes M) \quad ; \quad C^*(L,\mathfrak{g}; M) = \mathrm{Hom}_{\mathfrak{g}}(\Lambda^*(L/\mathfrak{g}), M)$$

with differentials (1) and (2) respectively ([F]). One may, with the obvious changes, give the analogous definitions for the cases when L is a Lie superalgebra ([Le]), or a differential graded algebra ([Q]), or a topological algebra ([F]). If $M = \mathbb{C}$ with trivial action of L then we put $H_*(\mathfrak{g}, M) = H_*(\mathfrak{g})$ etc.

Now we shall define the Weyl algebra of L (cf. [F]). Let $\mathbb{C}[\varepsilon]$ be the free skew commutative graded algebra with generator ε, deg $\varepsilon = 1$. Denote by $L[\varepsilon]$ the differential graded Lie algebra $L \otimes \mathbb{C}[\varepsilon]$ with differential acting as follows: $d(\ell \otimes \varepsilon) = \ell \otimes 1$; $d(\ell \otimes 1) = 0$. Put

$$W^*(L) = C^*(L[\varepsilon]); \quad W_*(L) = C_*(L[\varepsilon]).$$

The complex W^* is called a Weyl algebra of L. It is clear that W^*, W_* are contractible. If $\mathfrak{g}$ is a subalgebra reductive in L then we put

$$W^*(L,\mathfrak{g}) = C^*(L[\varepsilon], \mathfrak{g} \otimes 1); \quad W_*(L, \mathfrak{g}) = C_*(L[\varepsilon], \mathfrak{g} \otimes 1).$$

One has the projection

$$W^*(L, \mathfrak{g}) \to W^*(\mathfrak{g}, \mathfrak{g})$$

which is clearly a cohomology isomorphism. Thus,

$$H^{2k}(W^*(L, \mathfrak{g})) \xrightarrow{\sim} S^k(\mathfrak{g}^*)^{\mathfrak{g}}; \quad H^{2k+1}(W^*(L, \mathfrak{g})) = 0.$$

If $\rho : \mathfrak{g} \to L$ is a Lie algebra homomorphism such that $\rho(\mathfrak{g})$ is reductive in L then one has a characteristic homomorphisms

$$H^{2k}(W^*(L, \rho(\mathfrak{g}))) \leftarrow S^k(\mathfrak{g}^*)^{\mathfrak{g}}.$$

It is clear that

$$W^*(L,\mathfrak{g}) = \oplus W^{i,2n}(L, \mathfrak{g}) = \oplus C^i(L, \mathfrak{g}; S^n \mathfrak{g}^*);$$

if d is the differential in W^* then $d = d_1 + d_2$, $d_1 : W^{i,2n} \to W^{i+1,2n}$; $d_2 : W^{i,2n} \to W^{i-1,2(n+1)}$; d_1 is the differential (1). Thus, there is a spectral sequence $E_1^{p,2q} = H^p(L, \mathfrak{g}; S^q L^*) \Rightarrow H^{p+q}(W^*(L, \mathfrak{g}))$. Similarly for the absolute case.

Now recall the basic definitions on the Hochschild and cyclic homology. Let A be an associative algebra. Then Hochschild homology of A is the homology of the complex $C_*(A)$:

$$C_k(A) = A^{\otimes(k+1)}; \quad \delta : C_k(A) \to C_{k-1}(A);$$

$$\delta(a_0 \otimes \ldots \otimes a_k) = a_1 \otimes \ldots \otimes a_k a_0 + \sum_{i=1}^{k} (-1)^i a_0 \otimes \ldots \otimes a_{i-1}a_i \otimes \ldots \otimes a_k$$

This homology is denoted by $HH_*(A)$; one has

$$HH_*(A) \xrightarrow{\sim} \mathrm{Tor}_*^{A \otimes A^o}(A, A),$$

(cf. [CE]), where A^o is the algebra opposite to A. Put

$$\tau(a_o \otimes \ldots \otimes a_k) = (-1)^n\, a_1 \otimes \ldots \otimes a_k \otimes a_o;$$

$$HC_*(A) = H_*(C_*(A)/\mathrm{im}(1 - \tau)).$$

This is the cyclic homology of A ([C], [FT]). It is related to Lie algebra homology by the following ([LQ], [FT]):

$$H_*(\mathfrak{gl}(A)) \xrightarrow{\sim} S^*(HC_{*-1}(A)), \tag{3}$$

where $\mathfrak{gl}(A)$ is the Lie algebra of finite matrices with coefficients in A.

One may define the Hochschild cohomology HH^* to be the cohomology of the complex dual to $C_*(A)$ and the continuous cohomology HH_c^* of topological algebras. One may also define the Hochschild and cyclic homology of superalgebras and differential graded algebras so that the isomorphism (3) holds (cf. [B]).

1.2. <u>Generalized characters</u>. Let L be a Lie algebra and A an associative algebra; assume that π is a Lie algebra homomorphism from L to A. The map π determines the homomorphism $U(L) \to A$ of associative algebras and the induced homomorphism $HH_*(U(L)) \longrightarrow$ $\longrightarrow HH_*(A)$. It is easy to see ([CE]) that $HH_*(U(L))$ is isomorphic to the Lie algebra homology of L with coefficients in U(L) with the action $\ell \cdot u = \ell u - u\ell$, $\ell \in L$, $u \in U(L)$. The module U(L) is isomorphic to $S^*(L)$. Thus, we obtain a set of mappings

$$\chi_i^k(\pi) : H_i(L, S^k(L)) \to HH_i(A). \tag{4}$$

They are analogous to the classical invariant polynomials and to the characters of finite dimensional representations. To explain this, recall that if $A = M_N(C)$ then the unique nontrivial characters (4) are the mappings

$$\chi_0^k(\pi) : H_0(L, S^k L) \to HH_0(A) \xrightarrow{\sim} \mathbb{C} ;$$

the elements of $\mathrm{Hom}_{\mathbb{C}}(H_0(L, S^k L); \mathbb{C})$ are the invariant polynomials of degree k on L. The character acts as follows:

$$\chi_0^k(\pi)(\ell) = \mathrm{tr}(\pi(\ell)^k), \qquad \ell \in L.$$

Now let A be such that $HH_i(A) = 0$ for all $i \neq n$ and $HH_n(A) \xrightarrow{\sim} \mathbb{C}$ where n is the fixed non-negative integer.

Examples. 1) $A = M_N(\mathbb{C})$; $n = 0$.

2) Let V be an infinite dimensional vector space, End V the algebra of all linear operators $V \to V$ and J the ideal of End V consisting of all operators with finite-dimensional range. Put $I = \mathrm{End}\, V/J$. Then $HH_1(I) \xrightarrow{\sim} \mathbb{C}$ and $HH_i(I) = 0$, $i \neq 1$.

3) $HH_n(I^{\otimes n}) \xrightarrow{\sim} C$; $HH_i(I^{\otimes n}) = 0$, $i \neq n$. This follows from the Kunneth isomorphism for HH_* ([CE]).

4) Let Diff_n be the algebra of differential operators in $\mathbb{C}^n$ with polynomial coefficients. Then $H_{2n}(\mathrm{Diff}_n) \xrightarrow{\sim} \mathbb{C}$, $HH_i(\mathrm{Diff}_n) = 0$, $i \neq 2n$ (cf. §3).

Proposition 1.2.1. 1) The cohomology $H^*(\mathfrak{gl}(A))$ is the free skew commutative graded algebra with the generators $\eta_{n+1}, \eta_{n+3}, \eta_{n+5}, \ldots$, where $\eta_\alpha \in H^\alpha$.

2) The cohomology $H^*(\mathfrak{gl}(A), S^*\mathfrak{gl}(A)^*)$ (which is the first term of the spectral sequence converging to $H^*(W^*)$) is the free skew commutative algebra with generators η_{n+2k+1}, $k \geqslant 0$, and $\xi_k \in H^n(\mathfrak{gl}(A), S^k\mathfrak{gl}(A)^*)$, $k > 0$. (The differentials in the spectral sequence map η to ξ and ξ to zero.)

Proof. The statement 1) follows from (3) and from the fact that $HC_{n+2i}(A) = \mathbb{C}$, $i \geqslant 0$, and $HC_j(A) = 0$ elsewhere (which may be deduced from [FT], Th. 1.2.4). The proof of 2) (with the technical refinement which we shall need below) contains in §3. ∎

Let A be a topological algebra. The main example for us is the algebra of differential operators on $\mathbb{C}^n$ (we shall also denote it by Diff_n) whose coefficients are the formal series in n variables. The topology is induced by the $\mathfrak{m}$-adic topology on $\mathbb{C}[x_1, \ldots, x_n]$ where $\mathfrak{m}$ is the maximal ideal of the origin. Then it may be easily shown that $HH_c^{2n}(A) \xrightarrow{\sim} \mathbb{C}$, $HH_c^i(A) = 0$, $i \neq 2n$, and that Proposition 1.2.1 holds for the continuous Lie algebra cohomology of $\mathfrak{gl}(A)$.

Let π be the natural representation of $\mathfrak{gl}(A)$ in $M_\infty(A)$ (i.e. in the associative algebra of finite matrices over A). The characters

$$\chi_n^k(\pi) : H_n(\mathfrak{gl}(A), S^k\, \mathfrak{gl}(A)) \to HH_n(A) \to \mathbb{C} \quad .$$

are the elements of $H^n(\mathfrak{gl}(A), S^k(\mathfrak{gl}(A)^*))$. It may be shown that $\chi_n^k(\pi) = \xi_k$.

1.3. Geometric constructions.

Let M be a nonsingular complex manifold. Consider, following [F], an infinite-dimensional manifold $\widetilde{M}$ of all formal coordinate systems on M. A point of $\widetilde{M}$ is a couple (m, f) where $m \in M$ and f is an ∞-jet of a map $U \to \mathbb{C}^n$ where U is a neighbourhood of m in m, $f(m) = 0$ and the Jacobian of f in m is nonzero. It is clear that $\widetilde{M}$ is a projective limit of finite-dimensional complex manifolds. There is an action of the Lie algebra W_n on $\widetilde{M}$. Recall that W_n consists of vector fields $\sum_{1 \le i \le n} f_i \partial_{x_i}$ where f_i are formal power series in n variables. Introduce the $\mathfrak{m}$-adic topology on W_n. Throughout this section we shall regard all the objects connected with W_n equipped with the topology. In particular, the Weyl algebra of W_n is by definition the complex of continuous cochains of the differential graded topological Lie algebra $W_n[\varepsilon]$.

The action of W_n on $\widetilde{M}$ determines the structure of a principal homogeneous space on $\widetilde{M}$. This means that there is a W_n-valued one=

form Ω such that $d\Omega + \frac{1}{2}[\Omega,\Omega] = 0$ (the Maurer-Cartan equation) and that for any point $s \in \widetilde{M}$ the map $\Omega_s : T_s \to W_n$ is an isomorphism (where T_s is the tangent space to $\widetilde{M}$ in s).

The Lie algebra W_n contains a subalgebra of linear vector fields of the form $\sum a_{ij}x_i\partial_{x_j}$, $a_{ij} \in \mathbb{C}$, which is isomorphic to $\mathfrak{gl}_n(\mathbb{C})$ (or simply $\mathfrak{gl}_n$). The action of $\mathfrak{gl}_n$ on $\widetilde{M}$ is integrable to the action of the group $GL_n(C)$. The quotient space $\widetilde{M}/GL_n(C)$ is homotopically equivalent to M.

Let $\pi : S \to N$ is a bundle whose fibers are nonsingular n-dimensional compact complex manifolds (N and S are nonsingular). We shall construct the bundle $\widetilde{\pi} : \widetilde{S} \to N$. A point of $\widetilde{S}$ is a couple (s, f) where $s \in S$ and f is an ∞-jet of a holomorphic map $U \to \mathbb{C}^n$ where U is a neighbourhood of s in the fiber of π and $f(s) = 0$, f nondegenerate in s. The projection $\widetilde{\pi}$ maps (s, f) to $\pi(s)$. It is clear that for $n \in N$ $\widetilde{\pi}^{-1}(n) = \widetilde{\pi^{-1}(n)}$.

The fibres of $\widetilde{\pi}$ are the principal homogeneous spaces. This means that for any fiber there is a W_n-valued form on it which satisfies the Maurer-Cartan equation. We define a connection on S to be a W_n-valued 1-form which is invariant under the natural action of W_n and coincides with Ω on every fiber. It is easy to show that such a form does exist.

A connection determines a homomorphism from the Weyl algebra $W^*(W_n)$ to the de Rham complex $\Omega^*_{\widetilde{S}}$ of the manifold $\widetilde{S}$. The relative Weyl algebra $W^*(W_n, \mathfrak{gl}_n)$ maps into $\Omega^*_{\widetilde{S}/GL_n}$. Note that the spectral sequence converging to $H^*(W^*(W_n))$ (resp. $H^*(W^*(W_n, \mathfrak{gl}_n))$) maps into the Leray spectral sequence of the fibration $\widetilde{S} \to N$ (resp. $\widetilde{S}/GL_n \to N$). In particular, $E_1^{p,2q} \simeq E_2^{p,2q} \simeq H^p(W_n, \mathfrak{gl}_n; S^qW_n^*)$ maps into $H^{2q}(N, H^p(\overline{F}))$ where $\overline{F}$ is the fiber of the fibration $\widetilde{S}/GL_n \to$ $\to N$. Note that $\overline{F}$ is homotopically equivalent to the fiber F of the fibration $S \to N$. For $p = 2n$, $H^{2n}(F) \to \mathbb{C}$. Thus, we have constructed the homomorphisms

$$H^{2n}(W_n, \mathfrak{gl}_n; S^q W_n^*) \longrightarrow H^{2q}(N). \tag{5}$$

Remark 1.3.1. The above construction is analogous to Weyl's definition of characteristic classes. Indeed, let G be a semisimple Lie group and ξ a G-fibration with base N. The Weyl homomorphism is the map $H^o(\mathfrak{g}, S^q\mathfrak{g}^*) \to H^{2q}(N)$. In our case, the elements of $H^o(\mathfrak{g}, S^q\mathfrak{g}^*)$, i.e., the invariant polynomials on , are replaced by the elements of $H^{2n}(W_n, \mathfrak{gl}_n; S^q W_n^*)$. Now we shall describe the general situation.

Let L be a Lie algebra, $E \to N$ a fibration with the fiber F, L acts on E and the fibers are principal homogeneous L-spaces. Then one may define a connection form Ω on E. Let $\rho : L \to \mathfrak{A}$ be a Lie algebra homomorphism. The composition $\rho \circ \Omega$ is an $\mathfrak{A}$-valued connection form on E. This form determines a map from $W^*(\mathfrak{A})$ to Ω^*_E which induces the morphism of spectral sequences and thus the maps

$$H^p(\mathfrak{A}, S^q\mathfrak{A}^*) — H^{2q}(N, H^pF).$$

If L contains a subalgebra $\mathfrak{h}$ whose action is integrable to the action of a Lie group H then one may construct the following characteristic homomorphisms:

$$H^p(\mathfrak{A}, \rho(\mathfrak{h}); S^q\mathfrak{A}^*) \to H^{2q}(N, H^p(F/H)). \tag{6}$$

Now let A be an associative topological algebra such that the continuous Hochschild cohomology is concentrated in dimension $2n$ and $HH_c^{2n}(A) \simeq \mathbb{C}$. Let ρ be a continuous homomorphism $W_n \to \mathfrak{gl}(A)$, such that $\rho(\mathfrak{gl}_n)$ is reductive in $\mathfrak{gl}(A)$. The above constructions give the following mappings for any fibration $\widetilde{F} \to S \to N$ where S and N are compact complex manifolds:

$$\varphi_q(\rho) : \mathbb{C} \to H_c^{2n}(\mathfrak{gl}(A), \rho(\mathfrak{gl}_n); S^q\, \mathfrak{gl}(A)^*) \xrightarrow{\rho^*} \xrightarrow{\rho^*} H_c^{2n}(W_n, \mathfrak{gl}_n; S^q W_n^*) \to H^{2q}(N). \tag{7}$$

(The left isomorphism follows from Proposition 1.2.1 and from the Hochschild-Serre spectral sequence; see §3 for more detail.)

Definition 1.3.2. Set

$$\mathrm{ch}(\rho) = \sum_{q=0}^{\infty} (-1)^q\ \varphi_q(\rho)\ (1)/q! \in H^{**}(N)$$

(here and below we write H^{**} for $\prod_{q \geq 0} H^q$).

So, we have put in correspondence to a representation of W_n in A the distinguished elements $\varphi_q(\rho)(1)$ in every even cohomology group. Our next aim is to relate these elements to the characteristic classes.

Let $S \to N$ be as above. Let G be a complex Lie group and $\bar{\pi} : P \to S$ - holomorphic G-bundle. Define following [F] an infinite-dimensional manifold $\tilde{P}$. A point of $\tilde{P}$ is a couple (s, f) where $s \in S$ and f is defined as follows. Let U be a neighbourhood of s in the fiber of $S \to N$ and U_1 a neighbourhood of the origin in $\mathbb{C}^n$; then f is an ∞-jet in $\bar{\pi}^{-1}s$ of a morphism $\bar{\pi}^{-1}U \to U_1 \times G$ which is nondegenerate in $\bar{\pi}^{-1}s$ and commutes with the action of G. In other words, f is a formal trivialization of the restriction of $\bar{\pi}$ to the fiber of $S \to N$ together with the formal coordinate system in the fiber. The map $p : \tilde{P} \to N$, $p(s, f) = \bar{\pi}(s)$, turns $\tilde{P}$ to be a bundle whose fibers are principal homogeneous spaces over a Lie algebra which we shall now describe.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}(\mathcal{O}_n) = \mathfrak{g} \otimes \mathbb{C}[[x_1, \dots, x_n]]$ the Lie algebra of $\mathfrak{g}$-valued formal power series with the commutation law

$$[g_1 \otimes a_1, g_2 \otimes a_2] = [g_1, g_2] \otimes a_1 a_2, \quad g_i \in \mathfrak{g}, \quad a_i \in \mathbb{C}[[x_1, \dots, x_n]]$$

The Lie algebra W_n acts on $\mathfrak{g}(\mathcal{O}_n)$ by derivations, and we denote by $W_n \ltimes \mathfrak{g}(\mathcal{O}_n)$ the semidirect product of W_n and $\mathfrak{g}(\mathcal{O}_n)$. This algebra contains a subalgebra $\mathfrak{gl}_n \oplus \mathfrak{g}$, $\mathfrak{gl}_n \subset W_n$, $\mathfrak{g} \simeq \mathfrak{g} \otimes 1 \subset \mathfrak{g}(\mathcal{O}_n)$. Let A be, as above, a topological algebra whose Hochschild cohomology is concentrated in dimension $2n$ and $HH_c^{2n}(A) = \mathbb{C}$; let $\rho : W_n \ltimes \mathfrak{g}(\mathcal{O}_n) \to \mathfrak{gl}(A)$ be a Lie algebra homomorphism. Then one may, as above, obtain the following maps:

$$\varphi_q(\rho) : \mathbb{C} \to H_c^{2n}(\mathfrak{gl}(A), \rho(\mathfrak{gl}_n \oplus \mathfrak{g}); S^q(\mathfrak{gl}(A)^*)) \to$$

$$\to H_c^{2n}(W_n \ltimes \mathfrak{g}(\mathcal{O}_n), \mathfrak{gl}_n \oplus \mathfrak{g}; S^q(W_n \ltimes \mathfrak{g}(\mathcal{O}_n))^*) \to H^{2q}(N).$$

Put, as in Definition 1.2.2,

$$\mathrm{ch}(\rho) = \sum (-1)^q \varphi_q(\rho)(1)/q! \tag{7'}$$

Let Q be a finite-dimensional representation of $\mathfrak{g}$. It is clear that $W_n \ltimes \mathfrak{g}(\mathcal{O}_n)$ acts on the space $Q \otimes \mathbb{C}[[x_1, \ldots, x_n]]$. So we obtain the map

$$W_n \ltimes \mathfrak{g}(\mathcal{O}_n) \to \mathfrak{gl}_{\dim Q}(\mathrm{Diff}_n) \to \mathfrak{gl}(\mathrm{Diff}_n).$$

Denote the composition by $\rho(Q)$. Furthermore, let λ be a finite-dimensional representation of $\mathfrak{gl}_n$; it determines the representation of W_n in the space of formal tensor fields of corresponding type. This provides a homomorphism

$$\rho_\lambda : W_n \to \mathfrak{gl}_{\dim \rho}(\mathrm{Diff}_n) \hookrightarrow \mathfrak{gl}(\mathrm{Diff}_n).$$

Theorem 1.3.3.

$$\mathrm{ch}\,\rho(Q) = \pi_*(\mathrm{ch}\,\mathcal{E}(Q) \cdot \mathrm{td}\,\mathcal{T}_{S/N}) \tag{8}$$

$$\mathrm{ch}\,\rho_\lambda = \pi_*(\mathrm{ch}\,\mathcal{T}^\lambda_{S/N} \cdot \mathrm{td}\,\mathcal{T}_{S/N}) \tag{9}$$

where π_* is the transfer in cohomology, $\mathcal{E}(Q)$ is the vector bundle associated to the representation Q, $\mathcal{T}^\lambda_{S/N}$ is the relative bundle of tensor fields of type λ and $\mathcal{T}_{S/N}$ is the relative tangent bundle.

Our further plan in the following. In §2 we shall represent the left hand sides in (8), (9) as the transfers of the elements of H^*S which are the images of certain cohomology classes of the Weyl algebras under the characteristic homomorphisms (5). Furthermore, we shall formulate the theorem which express these classes in terms of the characteristic classes. This latter result is a purely algebraic theorem about the Lie algebra cohomology which shall be discussed in detail in §§ 4, 5. In §3 we state and prove some technical results on Hochschild cohomology and Lie algebra cohomology.

§2. Algebraic formulation of the main theorem

2.1. The universal cohomology classes of relative Weyl algebras. Let A be an associative algebra such that $HH_n(A) \xrightarrow{\sim} \mathbb{C}$ and $HH_i(A) = 0$, $i \neq n$. Assume $n > 0$. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(A)$ a homomorphism such that $\rho(\mathfrak{g})$ is reductive in $\mathfrak{gl}(A)$. Our aim is to define the distinguished cohomology classes in $H^{n+2q}(W_*(\mathfrak{gl}(A), \rho(\mathfrak{g}))$.

Let $(L, \mathfrak{g})$ be a pair consisting of a Lie algebra L and a subalgebra $\mathfrak{g}$ reductive in L. For any integer j, define a subcomplex $W_*(L, \mathfrak{g}; j)$ in $W_*(L, \mathfrak{g})$. Recall from 1.1 that $W_* = \oplus W_{p,2q}$ and $d = d_1 + d_2$, $d_1 : W_{p,2q} \to W_{p-1,2q}$; $d_2 : W_{p,2q} \to W_{p+1,2(q-1)}$. Put

$$W_*(L, \mathfrak{g}; j) = \bigoplus_{p>j} W_{p,*} \oplus \mathrm{Im}(d_1 : W_{j+1,*} \longrightarrow W_{j,*});$$

$$W_*^{(j)}(L, \mathfrak{g}) = W_*(L, \mathfrak{g})/W_*(L, \mathfrak{g}; j).$$

Lemma 2.1.1. Assume that $H_i(L, \mathfrak{g}; S^qL) = 0$ for all $i < j$ and $q > 0$. Then

$$H_i(W_*^{(j)}(L, \mathfrak{g})) \xrightarrow{\sim} H_i(L, \mathfrak{g}), \quad i \leqslant j;$$

$$H_{j+2q}(W_*^{(j)}(L,\mathfrak{g})) \xrightarrow{\sim} H_j(L,\mathfrak{g}\,;\,S^qL), \qquad q>0;$$

$$H_{j+2q+1}(W_*^{(j)}(L,\mathfrak{g})) = 0, \qquad q \geqslant 0.$$

Proof. This follows immediately from the spectral sequence converging to $H_*(W_*^{(j)}(L,\mathfrak{g}))$. ∎

Thus, we get the maps

$$H_{j+2q}(W_*(L,\mathfrak{g})) \longrightarrow H_j(L,\mathfrak{g};\ S^qL)$$

where j is the minimal dimension in which $H_*(L,\mathfrak{g};\ S^{>0}(L)) \neq 0$. We also have the dual maps for cohomology.

Now, let A be an associative algebra such that $HH_{2n}(A) \xrightarrow{\sim} \mathbb{C}$ and $HH_i(A) = 0$, $i \neq n$; $n > 0$; let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(A)$ be a homomorphism such that $\rho(\mathfrak{g})$ is reductive in $\mathfrak{gl}(A)$. Then the above construction together with Proposition 1.2.1 (cf. also Proposition 3.1.1) provides the homomorphisms

$$H_{2n+2q}(W_*(\mathfrak{gl}(A),\ \rho(\mathfrak{g})) \longrightarrow H_{2n}(\mathfrak{gl}(A),\ \rho(\mathfrak{g});\ S^q\,\mathfrak{gl}(A)^*)$$

and, dually, (5)

$$H^{2n+2q}(W^*(\mathfrak{gl}(A),\ \rho(\mathfrak{g})) \longleftarrow H^{2n}(\mathfrak{gl}(A),\ \rho(\mathfrak{g});\ S^q\,\mathfrak{gl}(A)^*) \xleftarrow{\sim} \mathbb{C}$$

On the other hand (cf. 1.1), there is a map

$$H^{2m}(W^*(\mathfrak{gl}(A),\ \rho(\mathfrak{g})) \longrightarrow S^m(\mathfrak{g}^*)^{\mathfrak{g}}. \qquad (6)$$

Within our approach, the Riemann-Roch problem is the problem of expressing of the distinguished elements given by (5) in terms of the homomorphism (6).

Before discussing this, we should like to construct the maps analogous to (5) in lower dimensions, i.e., for H^{2i} where $i \leqslant n$.

Let $\mathfrak{a} \simeq \mathbb{C}$ be the one-dimensional Abelian Lie algebra. Define the representation θ of $\mathfrak{g} \oplus \mathfrak{a}$ as follows:

$$\theta(g,\alpha) = \rho(g) + \alpha\cdot 1, \qquad g \in \mathfrak{g},\ \alpha \in \mathfrak{a}\ .$$

Replacing $\mathfrak{g}$ by $\mathfrak{g}+\mathfrak{a}$ in formulas (5), (6), we obtain the maps

$$\varphi_{q+n} : \mathbb{C} \to S^{n+q}((\mathfrak{g}\oplus\mathfrak{a})^*)^{\mathfrak{g}\oplus\mathfrak{a}} \to \bigoplus_{j=0}^{n+q} S^j(\mathfrak{g}^*)^{\mathfrak{g}}, \quad q > 0 \qquad (7)$$

Let φ^j_{q+n} be the homogeneous component of degree j in φ_{q+n}

<u>Lemma 2.1.2</u>. For any q, $\varphi^j_{q+n} = \varphi^j_{q+n+1}$.

<u>Proof</u>. This follows immediately from the definition of φ^j_{q+n} (cf. 4.1 for more detail). ∎

Put

$$\chi(\rho) = \sum_{j\geq 0} (-1)^j(\varphi^j/j!)(1) \in \prod_{j\geq 0} S^j(\mathfrak{g}^*)^{\mathfrak{g}},$$

where $\varphi^j = \varphi^j_{n+q}$, $q \gg 0$.

Thus, for a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(A)$ we have constructed its character which is an invariant formal series on $\mathfrak{g}$. Let A_1, A_2 be two algebras such that

$$HH^*(A_1) = HH^{2n}(A_1) \xrightarrow{\sim} \mathbb{C}; \quad HH^*(A_2) = HH^{2m}(A_2) \xrightarrow{\sim} \mathbb{C}.$$

Then, by Kunneth isomorphism, $HH^*(A_1 \otimes A_2) = HH^{2(n+m)}(A_1 \otimes A_2) \xrightarrow{\sim} \mathbb{C}$. For $\rho_i : \mathfrak{g} \to \mathfrak{gl}(A_i)$ one may define

$$\rho_1\otimes\rho_2 : \mathfrak{g} \to \mathfrak{gl}(A_1\otimes A_2),$$

$$(\rho_1\otimes\rho_2)(g) = \rho_1(g)\otimes 1 + 1 \times \rho_2(g).$$

Then $\chi(\rho_1\otimes\rho_2) = \chi(\rho_1)\cdot\chi(\rho_2)$. If o_1, o_2 - two representations of in A, then $\rho_1\oplus\rho_2$ is a representation:

$$(\rho_1\oplus\rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

and

$$\chi(\rho_1\oplus\rho_2) = \chi(\rho_1) + \chi(\rho_2).$$

2.2. <u>Riemann-Roch theorem for Lie algebra cohomology</u>.

Here and below we denote $\mathfrak{gl}(\mathrm{Diff}_n)$ by D_n. Consider, as in §1, the homomorphism $\mathfrak{gl}_n\oplus\mathfrak{gl} \xrightarrow{\rho} D_n$ which is the composition $\mathfrak{gl}_n\oplus\mathfrak{gl} \hookrightarrow W_n \ltimes \mathfrak{gl}(\mathcal{O}_n) \hookrightarrow D_n$. Identify $S^*(\mathfrak{gl}_n\oplus\mathfrak{gl})^{\mathfrak{gl}_n\oplus\mathfrak{gl}}$ with

$H^*(BGL_n \times BGL)$. Let τ_n, τ be universal vector bundles over BGL_n, BGL respectively. Put $\mathcal{T} = \tau_n \boxtimes 1$; $\mathcal{E} = 1 \boxtimes \tau$. Put in correspondence to the character of the representation ρ the element $\chi(\rho) \in HH^*(BGL_n \times BGL)$.

<u>Theorem 2.2.1</u>. $\chi(\rho) = \mathrm{ch}\, \mathcal{E} \cdot \mathrm{td}\, \mathcal{T}$.

An analogous statement may be easily formulated for the character of the representation $\mathfrak{gl}_n \to W_n \to D_n$ corresponding to the representation of W_n in the space of tensor fields.

<u>2.3. Relation to §1</u>. Let $\widetilde{P} \to N$ be, as in §1, the fibration whose fibers are principal $W_n \ltimes \mathfrak{gl}(\mathcal{O}_n)$ - homogeneous spaces. The connection Ω determines a map

$$W^*(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}) \to W^*(W_n \ltimes \mathfrak{gl}(\mathcal{O}_n), \mathfrak{gl}_n \oplus \mathfrak{gl}) \to \Omega^*_{\widetilde{P}/GL_n \times GL}$$

and the map

$$\Phi : S(\mathfrak{gl}_n^* \oplus \mathfrak{gl}^*)^{\mathfrak{gl}_n \oplus \mathfrak{gl}} \xrightarrow{\sim} H^{2*}(W^*(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl})) \to H^{2*}(\widetilde{P}/(GL_n \times GL))$$

It is easy to see from the definitions that the element $\mathrm{ch}(\rho) \in H^{**}(N)$ from the formula 7' of 1.2 is equal to $\pi_* \Phi(\chi(\rho))$. Thus, to deduce Theorem 1.3.3 from Theorem 2.2.1 it suffices to show that Φ is the Chern-Weyl homomorphism of the fibration $\widetilde{P} \to \widetilde{P}/(GL_n \times GL)$. Denote $L = W_n \ltimes \mathfrak{gl}(\mathcal{O}_n)$, $\mathfrak{g} = \mathfrak{gl}_n \oplus \mathfrak{gl}$. Consider a $\mathfrak{g}$-valued connection form on L, i.e., a $\mathfrak{g}$-equivariant projection operator $\theta : L \to \mathfrak{g}$. Put $\Theta(X, Y) = \theta([X, Y]) - [\theta(X), \theta(Y)]$. Define a homomorphism of differential graded algebras

$$\Psi : W^*(\mathfrak{g}) \longrightarrow W^*(L).$$

We need only define Ψ on the generators

$$(\ell : \mathfrak{g} \to \mathbb{C}) \in W^1; \qquad (\lambda : \mathfrak{g}\varepsilon \to \mathbb{C}) \in W^2.$$

Put

$$(\Psi\ell)(X) = \ell(\theta(X)); \ (\Psi\lambda)(X \wedge Y + \varepsilon Z) = -\lambda(\varepsilon\Theta(X, Y)) + \lambda(\varepsilon\theta(Z)).$$

It is easy to see that Ψ is well defined and that the induced map $W^*(\mathfrak{g}, \mathfrak{g}) \to W^*(L, \mathfrak{g})$ is a quasi-isomorphism which is cohomology inverse to the characteristic homomorphism of 1.1. On the other hand, let $\mathfrak{L}$ be a connection form on $\widetilde{P}$. Then $\theta\circ\mathfrak{L}$ is the $(\mathfrak{gl}_n\oplus\mathfrak{gl})$-valued connection in the fibration $\widetilde{P}\to\widetilde{P}/(GL_n\times GL)$. The direct verification shows that the composition

$$\mathfrak{L}^*_{\widetilde{P}} \leftarrow W^*(W_n\ltimes\mathfrak{gl}(\mathcal{O}_n)) \leftarrow W^*(\mathfrak{gl}_n\oplus\mathfrak{gl})$$

is exactly the Chern-Weyl homomorphism associated to the connection $\theta\circ\mathfrak{L}$. Thus, we have shown that Theorem 2.2.1 implies Theorem 1.3.3.

§3. Homology of the algebra of differential operators

3.1. Relation between Lie algebra homology and Hochschild homology. Throughout this subsection, A shall denote an associative algebra such that $HH_n(A)=\mathbb{C}$, $HH_i(A)=0$, $i\neq n$; $n>0$.

Let τ be a Hochschild cocycle representing the basis cohomology class of $HH^n(A)$. Set

$$\omega_\tau(x_1\cdot m_1,\dots,x_{n+1}\cdot m_{n+1}) = \sum_{\sigma\in S_n}\operatorname{sgn}\sigma\cdot\operatorname{tr}(m_{\sigma 1}\dots m_{\sigma n}\,m_{n+1})\,\tau(x_{\sigma 1},\dots,x_{\sigma n},x_{n+1})$$

for $x_i\in A$, $m_i\in\mathfrak{gl}(\mathbb{C})$. It is easy to verify that ω_τ is a cocycle of the standard complex $C^*(\mathfrak{gl}(A),\mathfrak{gl}(A)^*)$. Consider a map

$$\mu_*: S^*(\mathfrak{gl}(A))\to\mathfrak{gl}(A);$$

$$\mu_q(X_1\cdot\ \dots\ \cdot X_q) = \frac{1}{q!}\sum_{\sigma\in S_q} X_{\sigma 1}\dots X_{\sigma q},\quad X_i\in\mathfrak{gl}(A). \tag{1}$$

It is clear that μ_* is a homomorphism of modules over the Lie algebra $\mathfrak{gl}(A)$. Consider the dual homomorphisms $\mu_q^*: S^q(\mathfrak{gl}(A)^*)\leftarrow\mathfrak{gl}(A)^*$ and the induced homomorphisms

$$\mu_q^*: C^*(\mathfrak{gl}(A),\ S^q(\mathfrak{gl}(A)^*))\leftarrow C^*(\mathfrak{gl}(A),\mathfrak{gl}(A)^*).$$

Proposition 3.1.1. 1) For $q>0$, $H^n(\mathfrak{gl}(A),\ S^q\mathfrak{gl}(A)^*)\xrightarrow{\sim}\mathbb{C}$ and $H^i(\mathfrak{gl}(A),\ S^q\mathfrak{gl}(A)^*)=0$, $i<n$.

2) The cocycles $\mu_q^* \omega_\tau$ represent nonzero cohomology classes.

Proof. Let $\mathbb{C}[\varepsilon]$ denote a superalgebra with one generator and one relation $\varepsilon^2 = 0$. Let $A[\varepsilon] = A \otimes \mathbb{C}[\varepsilon]$. One has

$$H_*(\mathfrak{gl}(A[\varepsilon]), \mathbb{C}) \xrightarrow{\sim} \bigoplus_{q \geq 0} H_*(\mathfrak{gl}(A), S^q \mathfrak{gl}(A));$$

on the other hand

$$H_*(\mathfrak{gl}(A[\varepsilon]), \mathbb{C}) \xrightarrow{\sim} S^*(HC_{*-1}(A[\varepsilon])).$$

Compute the cyclic homology of the superalgebra $A[\varepsilon]$. One has

$$HH_i(\mathbb{C}[\varepsilon]) \xrightarrow{\sim} \mathbb{C}^2, \quad i \geq 0;$$

the basis in this space consists of the elements ω_i^1 and ω_i^2 represented by the cycles $\varepsilon \otimes \ldots \otimes \varepsilon \otimes 1$ and $\varepsilon \otimes \ldots \otimes \varepsilon$ respectively. Let B be the differential in Hochschild homology (cf. [FT]); then $B\omega_i^1 = 0$, $B\omega_i^2 = \omega_i^1$. From the spectral sequence converging to cyclic homology ([FT], Th. 1.2.) one sees that

$$HC_i(\mathbb{C}[\varepsilon])/HC_i(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}, \quad i \geq 0,$$

and that the generators in these spaces are ω_i^2. Now consider the analogous spectral sequence for $A[\varepsilon]$. Since the differential B is compatible with the Kunneth isomorphism, one has

$$HC_*(A[\varepsilon]) \xrightarrow{\sim} HC_{*-n}(\mathbb{C}[\varepsilon]);$$

$$HC_*(A[\varepsilon])/HC_*(A) \longrightarrow HC_{*-n}(\mathbb{C}[\varepsilon])/HC_{*-n}(\mathbb{C});$$

thus,

$$HC_i(A[\varepsilon]) = 0, \quad i < n; \quad HC_{i+n}(A[\varepsilon]) \xrightarrow{\sim} HC_{i+n}(A) \oplus \mathbb{C}, \quad i \geq 0;$$

the generators in these supplementary summands are the images of the elements $\alpha_n T \omega_i^2$ under the map $HH_* \to HC_*$. Here α_n is a generator in $HH_n(A)$ and T is the exterior multiplication in Hochschild homology (cf. [CE]). This proves the statement 1) of the Proposition (and also Proposition 1.2.1). The statement 2) follows immediately from the explicit form of the isomorphism (1) (cf. [LQ], [FT]). To prove 3) note that if α is a cycle of $C_*(\mathfrak{gl}(A), \mathfrak{gl}(A))$ and $\omega_\tau(\alpha) \neq 0$ then $\alpha \cdot 1^{q-1}$ is a cycle of $C_*(\mathfrak{gl}(A), S^q\mathfrak{gl}(A))$ and $(\mu_q^* \omega_\tau)(\alpha \cdot 1^{q-1}) = \omega_\tau(\alpha) \neq 0$. Thus, the cohomology class of $\mu_q^* \omega_\tau$ is nonzero for $q > 0$.

Let $\mathfrak{g}$ be reductive in $\mathfrak{gl}(A)$, $q > 0$.

Proposition 3.1.2. 1) $H^n(\mathfrak{gl}(A), \mathfrak{g}; S^q\mathfrak{gl}(A)^*) \xrightarrow{\sim} \mathbb{C}$;

$$H^i(\mathfrak{gl}(A), \mathfrak{g}; S^q\mathfrak{gl}(A)^*) = 0, \quad i < n.$$

2) Let ω be a generator in $H^n(\mathfrak{gl}(A), \mathfrak{g}; \mathfrak{gl}(A)^*)$. Then $\mu_q^*\omega$ generate $H^n(\mathfrak{gl}(A), \mathfrak{g}; S^q\mathfrak{gl}(A)^*)$.

Proof. Proposition 3.1.1 together with the Hochschild-Serre spectral sequence imply that

$$H^i(\mathfrak{gl}(A), \mathfrak{g}; S^q\mathfrak{gl}(A)^*) \xleftarrow{\sim} H^i(\mathfrak{gl}(A), S^q\mathfrak{gl}(A)^*), \quad i \leq n. \blacksquare$$

3.2. Hochschild homology of the algebra of differential operators.

Theorem 3.2.1. ([FT1]). $HH_{2n}(\mathrm{Diff}_n) \xrightarrow{\sim} C$; $HH_i(\mathrm{Diff}_n) = 0$, $i \neq 2n$.

Proof. In order to prove the Theorem and to find the explicit form of the Hochschild cocycle representing the unique nontrivial cohomology class of $HH^{2n}(\mathrm{Diff}_n)$ we shall use the Koszul resolution from [K]. Let $C_0 = C_2 = \mathrm{Diff}_1^{\otimes 2}$; $C_1 = \mathrm{Diff}_1^{\otimes 2} \oplus \mathrm{Diff}_1^{\otimes 2}$; $C_i = 0$, $i > 2$; $d_i : C_i \to C_{i-1}$, $i \geq 1$;

$$d_1(X_1 \otimes X_2, X_3 \otimes X_4) = (X_1\partial \otimes X_2 - X_1 \otimes \partial X_2) - (X_3 x \otimes X_4 - X_3 \otimes xX_4),$$

$$d_2(X_1 \otimes X_2) = (X_1 x \otimes X_2 - X_1 \otimes xX_2, X_1\partial \otimes X_2 - X_1 \otimes \partial X_2)$$

for $X_i \in \mathrm{Diff}_1$, here ∂, $x \in \mathrm{Diff}_1 \simeq \mathbb{C}[x, \partial]$, $x\partial - \partial x = 1$. It is clear that $d_{i-1}d_i = 0$ and that (C_*, d_*) is a free bimodule resolution of Diff_1. Thus,

$$HH_*(\mathrm{Diff}_1) \xrightarrow{\sim} H_*(C_* \otimes_{\mathrm{Diff}_1 \otimes \mathrm{Diff}_1^o} \mathrm{Diff}_1);$$

it is easy to see that the right hand side is isomorphic to $\mathbb{C}$ and concentrated in H_2. The basis element is represented by a cycle

$$1 \in \mathrm{Diff}_1 \xrightarrow{\sim} C_2 \times_{\mathrm{Diff}_1 \otimes \mathrm{Diff}_1^{\circ}} \mathrm{Diff}_1 .$$

This proves the Theorem for n = 1. The general case follows from the Kunneth isomorphism and from the fact that $\mathrm{Diff}_n \xrightarrow{\sim} \mathrm{Diff}_1^{\otimes n}$.

Corollary 3.2.2. $H^{2n}(D_n, D_n^*) \xrightarrow{\sim} \mathbb{C}$; $H^i(D_n, D_n^*) = 0$, $i < 2n$.

Proof. This follows from Proposition 3.1.1. ∎

Remark 3.2.3. Recently Brylinski and Getzler [BG] and Wodzicki [W] proved the isomorphism

$$HH_i(\mathrm{Diff}\ M) \xrightarrow{\sim} H_{DR}^{2\dim M - i}(M, \mathbb{C})$$

where M is an affine nonsingular algebraic manifold and Diff M is the ring of regular differential operators on M. The analogous statement holds when M is a C^∞-manifold.

3.3. The cocycles of the algebra of differential operators. Let τ be a Hochschild cocycle whose cohomology class generates $HH^2(\mathrm{Diff}_1)$. Let ω_τ be as in 3.2.

Lemma 3.3.1. There exists such 2-cocycle τ that

$$\omega_\tau(E_{11}(f\partial + \varphi),\ E_{11}(g\partial + \psi),\ E_{11}\cdot \sum_{k\geq 0} h_k \partial^k) =$$

$$= \sum_{k\geq 0}\left(\left(\frac{\varphi^{(k+1)}g - \psi^{(k+1)}f}{k+1} - \frac{f^{(k+2)}g - g^{(k+2)}f}{(k+1)(k+2)}\right) h_k\right)(0) \qquad (2)$$

for all $f, g, \varphi, \psi, h_k \in \mathbb{C}[x]$.

Proof. For any algebra A, let $B_*(A)$ be the bar resolution of the bimodule A:

$$B_n(A) = A^{\otimes(n+2)}; \qquad b : B_n(A) \to B_{n-1}(A);$$

$$b(a_{-1} \otimes \dots \otimes a_n) = \sum_{i=0}^{n} (-1)^i a_{-1} \otimes \dots \otimes a_{i-1}a_i \otimes \dots \otimes a_n$$

(cf. [CE]). Then the standard complex $C_*(A)$ (1.1) is isomorphic to

$B_*(A) \otimes_{A\otimes A^\circ} A$. We shall construct the chain map $\varphi_* : B_*(\mathrm{Diff}_1) \to$ $\to C_*$ where C_* is the Koszul resolution from 3.2. Then we shall define a cocycle τ to be the composition of $\varphi_2 \otimes_{\mathrm{Diff}_1\otimes \mathrm{Diff}_1^\circ} \mathrm{Diff}_1^\circ$ with the linear functional ℓ on $C_2 \otimes_{\mathrm{Diff}_1\otimes \mathrm{Diff}_1} \mathrm{Diff}_1 \xrightarrow{\sim} \mathrm{Diff}_1$ which sends $\sum h_k \partial^k$ to $h_o(0)$. The functional τ shall be a cocycle because ℓ is a 2-cocycle of the complex dual to $C_* \otimes_{\mathrm{Diff}_1\otimes \mathrm{Diff}_1} \mathrm{Diff}_1$.

We construct φ_* as follows. The homomorphism which puts in correspondence to an operator it's symbol is an isomorphism between Diff_1 and $C[X, \xi]$; identify $\mathrm{Diff}_1^{\otimes 2}$ and $\mathbb{C}[x, y, \xi, \eta]$ using this homomorphism. We have in C_* for $f,g \in \mathbb{C}[x, y, \xi, \eta]$:

$$d_2 f = ((\xi - \eta - \partial_y)f,\ (x - y + \partial_\xi)f);\quad d_1(f, g) = (x - y + \partial_\xi)f + (\xi - \eta - \partial_y)g).$$

Consider a complex C_*^o:

$$C_2^o = C_o^o = \mathbb{C}[x, y, \xi, \eta];\quad C_1^o = \mathbb{C}[x, y, \xi, \eta]^{\oplus 2};\quad d_*^o : C_*^o \to C_{*-1}^o;$$

$$d_2^o f = ((\xi - \eta)f,\ (x - y)f);\qquad d_1^o(f, g) = (y - x)f + (\xi - \eta)g.$$

It is easy to verify that the map $\exp(\partial_\xi \partial_y)$ provides an isomorphism $C_* \to C_*^o$. Indeed,

$$\left[\xi - \eta\ ,\ e^{\partial_\xi \partial_y}\right] = -\partial_y \cdot e^{\partial_\xi \partial_y};$$
$$\left[x - y\ ,\ e^{\partial_\xi \partial_y}\right] = \partial_\xi \cdot e^{\partial_\xi \partial_y}.$$

Put $C_{-1} = \mathrm{Diff}_1 \xrightarrow{\sim} C[x,\xi]$; $C_{-1}^o = C[x,\xi]$; $d_o(X_1 \otimes X_2) = X_1 X_2$; $(d_o^o f)(x,\xi) = f(x, x, \xi, \xi)$.

It is clear that the above isomosphism may be prolonged to an isomorphism of augmented complexes $C_* \to C_*^o$. This follows from the formula of symbol of product. The augmented complex admits a constructing homotopy $s_i : C_i^o \to C_{i+1}^o$, $i \geqslant -1$:

$$(s_{-1}f)(x, y, \xi, \eta) = f(x, \xi);\quad s_o f = (tf, t'f);\quad s_1(f, g) = tg;\quad s_i = 0,\ i > 1,$$

where

$$(tf)(x, y, \xi, \eta) = \frac{f(x, y, \xi, \eta) - f(x, y, \xi, \xi)}{\xi - \eta};$$

$$(t'f)(x, y, \xi, \eta) = \frac{f(x, y, \xi, \xi) - f(x, x, \xi, \xi)}{y - x};$$

direct verification shows that $s_{i-1}d_i^o + d_{i+1}^o s_i = 1_{C_i^o}$ for all i.

Now we shall construct, following to [CE], ch. , a chain map $\varphi = \bigoplus_{i \geq 0} \varphi_i$ using induction on i.

Put

$$\varphi_o\left(\sum g_k \partial^k \otimes \sum h_\ell \partial^\ell\right) = \sum_{k,\ell} g_k(x) h_\ell(y) \xi^k \eta^\ell.$$

Let $s' = \oplus s_i'$ be the constructing homotopy of the augmented complex C_*^o; $s_i' = e^{-\partial_\xi \partial_y} s_i e^{\partial_\xi \partial_y}$. Assume that the maps φ_j, $j < i$, are already constructed. Let $\alpha \in B_i(\mathrm{Diff}_1)$ be of the form $1 \otimes X_o \otimes \ldots \otimes X_{i-1} \otimes 1$. Put

$$\varphi_i(\alpha) = s_{i-1}' \varphi_{i-1} b\alpha;$$

for an arbitrary α we define $\varphi_i(\alpha)$ using $(\mathrm{Diff}_1) \otimes (\mathrm{Diff}_1^o)$ - linearity.

Proceeding in such a way we obtain for any operators X_o, X_1 with symbols f_o, f_1 respectively:

$$\varphi_1(1 \otimes X_o \otimes 1) = e^{-\partial_\xi \partial_y}\left[\frac{f_o(x,\xi) - f_o(x,\eta)}{\xi - \eta}, \frac{f_o(x,\xi) - f_o(y,\xi)}{y - x}\right];$$

$$\varphi_2(1 \otimes X_o \otimes X_1 \otimes 1) =$$

$$= e^{-\partial_\xi \partial_y}\left[\frac{e^{\partial_\xi \partial_y}\left(e^{-\partial_\xi \partial_y}\left(\frac{f_o(x,\xi) - f_o(y,\xi)}{y - x}\right) \cdot f_1(y,\eta)\right)\Big|_\xi^\eta}{\xi - \eta}\right]$$

(we use the notation $\Phi\Big|_\xi^\eta = \Phi(x, y, \xi, \eta) - \Phi(x, y, \xi, \xi)$).

Now let $X_o = f\partial + \varphi$, $X_1 = g\partial + \psi$, where $f, g, \varphi, \psi, \in C[x]$.

Put $\Delta f(x, y) = \frac{f(x) - f(y)}{x - y}$. We obtain from the formula for φ_2:

$$\varphi_2(1 \otimes X_0 \otimes X_1 \otimes 1) = \partial_y \Delta f \cdot g(y) - \Delta f \cdot g(y) \cdot \xi - \Delta \varphi \cdot g(y);$$

the image of the chain $X_0 \otimes X_1 \otimes \sum h_k \partial^k$ in $C_2 \otimes_{\mathrm{Diff}_1 \otimes \mathrm{Diff}_1^o} \mathrm{Diff}_1 \xrightarrow{\sim} \mathrm{Diff}_1$ has the symbol equal to

$$\sum_{k \geqslant 0} \lim_{y \to x} (\partial_x^k \partial_y \Delta f - \partial_x^k \Delta \varphi)(x, y) \cdot g(y) h_k(x) + (\ldots) \xi \quad ;$$

the proof of the Lemma follows now from the equalities

$$\lim_{y \to x} \partial_x^k \partial_y \Delta f = \frac{1}{(k+1)(k+2)} f^{(k+2)} \quad ;$$

$$\lim_{y \to x} \partial_x^k \Delta f = \frac{1}{k+1} f^{(k+1)} \quad . \quad \blacksquare$$

Remark 3.3.2. It is interesting to compare Lemma 3.3.1 with the computation in [ADKP] of the restriction of "Japanese 2-cocycle" to the algebra $\mathrm{Diff}_1^{\leq 1}(S^1)$.

Remark 3.3.3. It would be very important to find a satisfactory formula of the Hochschild 2-cocycle of Diff_1.

Let τ be a 2-cocycle of Diff_1 constructed in Lemma 3.3.1 and $\tau_n = \tau^{\otimes n}$ where $\otimes$ is the exterior multiplication $HH^*(A) \otimes HH^*(B) \xrightarrow{\sim} HH^*(A \otimes B)$ (dual to the comultiplication $HH_*(A \otimes B) \xrightarrow{\sim} HH_*(A) \otimes HH_*(B)$). The proof of Lemma 3.3.1 together with the implicit formula for $\otimes$ ([CE]) show that the expression

$$\omega_{\tau_n}(F\partial + \Phi \ , \ G\partial + \Psi \ , \ \sum H_k \partial^k)$$

where $F, G, \Phi, \Psi, H_k \in \mathfrak{gl}(\mathbb{C}[X_1, \ldots, X_n])$ depends only on $\partial^\alpha F(0)$, $\partial^\beta \Phi(0)$, ... where α, β are such multi-induces that $|\alpha| \neq 1$, $|\beta| \geqslant 1$. On the other hand, it is easy to see that ω_{τ_n} is $(\mathfrak{gl}_n \oplus \mathfrak{gl})$-invariant. Thus we obtain

Lemma 3.3.4. The cocycle ω_{τ_n} is an element of $C^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; D_n^*)$

whose cohomology class generates $H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; D_n^*)$. This class is determined by the equality:

$$\omega_{\tau_n}(E_{11}x_1, E_{11}\partial_{x_1}, \ldots, E_{11}x_n, E_{11}\partial_{x_n}, E_{11}) = 1. \qquad \blacksquare \tag{3}$$

§4. Relative local Riemann-Roch theorem

4.1. Construction of the character. The aim of the present subsection is to recall the basic construction of 2.1 and to make it somewhat more implicit using 3.1.

Let A be an associative algebra such that $HH_*(A) = HH_{2n}(A) \xrightarrow{\sim} \mathbb{C}$; let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(A)$ a homomorphism such that $\rho(\mathfrak{g})$ is reductive in $\mathfrak{gl}(A)$. Consider the homomorphisms

$$\mathbb{C} \xrightarrow{\sim} H^{2n}(\mathfrak{gl}(A), \rho(\mathfrak{g}); S^q \mathfrak{gl}(A)^*) \to$$
$$\to H^{2(n+q)}(W^*(\mathfrak{gl}(A), \rho(\mathfrak{g})) \to S^{n+q}(\mathfrak{g}^*)^{\mathfrak{g}} \tag{1}$$

for $q > 0$. The first homomorphism sends 1 to $\mu_q^* \omega$ where $[\omega]$ is the generator of $H^{2n}(\mathfrak{gl}(A), \rho(\mathfrak{g}); S^q\mathfrak{gl}(A)^*)$ (cf. 3.1); the second one is defined in 2.1; the first one is the characteristic map from 1.1. Denote the composition by $\varphi_{q+n}(\rho)$ or simply by φ_{q+n}. Thus, φ_* is determined up to a nonzero scalar.

Now define φ_j, $j \geq 0$, as follows. Let $\mathfrak{a}$ be a 1-dimensional Lie algebra with generator a. Consider the homomorphism $\theta : \mathfrak{g} \oplus \mathfrak{a} \to \mathfrak{gl}(A)$; $\theta(g, \alpha a) = \rho(g) + \alpha \cdot 1$, $g \in \mathfrak{g}$, $\alpha \in \mathbb{C}$. Applying the above construction to θ one obtains the maps $\mathbb{C} \to S^{n+q}((\mathfrak{g} \oplus \mathfrak{a})^*)^{\mathfrak{g} \oplus \mathfrak{a}}$, $q > 0$. For any $j \leq n+q$ there is a homomorphism $S^j(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{a^{n+q-j}} S^{n+q}(\mathfrak{g} \oplus \mathfrak{a})^{\mathfrak{g} \oplus \mathfrak{a}}$, $g_1 \ldots g_j \longmapsto g_1 \ldots g_j \cdot a^{n+q-j}$. Define $\varphi_j(\rho)$ to be the composition

$$\mathbb{C} \xrightarrow{\varphi_{n+q}(\Theta)} S^{n+q}((\mathfrak{g}\oplus\mathfrak{a})^*)^{\mathfrak{g}\oplus\mathfrak{a}} \xrightarrow{(\alpha^{n+q-j})^*} S^j(\mathfrak{g}^*)^{\mathfrak{g}}. \qquad (2)$$

Lemma 4.1.1. This map does not depend on q.

Proof. Consider the chain morphisms

$$\mathbb{1}^{\ell}: \; W_*(\mathfrak{gl}(A), \rho(\mathfrak{g})) \to W_{*+2\ell}(\mathfrak{gl}(A), \Theta(\mathfrak{g}\oplus\mathfrak{a}));$$

$$1^{\ell}: \; H_*(\mathfrak{gl}(A), \rho(\mathfrak{g}); S^*\mathfrak{gl}(A)) \to H_*(\mathfrak{gl}(A), \Theta(\mathfrak{g}\oplus\mathfrak{a}); S^{*+\ell}\mathfrak{gl}(A)).$$

It is easily seen from the definitions that the following diagram is commutative and that the vertical map on the right sends $\mu^*_{p+q}\omega'$ to $\mu^*_q\omega'$, whence the Lemma.

$$\begin{tikzcd}
S^{n+q+p}((\mathfrak{g}\oplus\mathfrak{a})^*)^{\mathfrak{g}\oplus\mathfrak{a}} \arrow[d, "(\alpha^p)^*"] & H^{2(n+p+q)}(W^*(\mathfrak{gl}(A), \Theta(\mathfrak{g}\oplus\mathfrak{a}))) \arrow[l] \arrow[d, "(\mathbb{1}^p)^*"] & H^{2n}(\mathfrak{gl}(A), \Theta(\mathfrak{g}\oplus\mathfrak{a}); S^{p+q}\mathfrak{gl}(A)^*) \arrow[l] \arrow[d, "(1^p)^*"] \\
S^{n+q}(\mathfrak{g}^*)^{\mathfrak{g}} & H^{2(n+q)}(W^*(\mathfrak{gl}(A), \rho(\mathfrak{g}))) \arrow[l] & H^{2n}(\mathfrak{gl}(A), \rho(\mathfrak{g}); S^q\mathfrak{gl}(A)^*) \arrow[l]
\end{tikzcd}$$

Now, let $A = \mathrm{Diff}_n$ and $\rho : \mathfrak{gl}_n \oplus \mathfrak{gl} \hookrightarrow \mathfrak{gl}(\mathrm{Diff}_n)$ be as in 1.3. We fix a generator in $H^{2n}(\mathfrak{gl}(A), \rho(\mathfrak{g}); \mathfrak{gl}(A))$ to be ω_{τ_n} satisfying (3) of 3.3. Thus, we have defined $\varphi_j(\rho)$, $j \geq 0$, implicitly. Put

$$\mathrm{ch}(\rho) = \sum_{j=0}^{\infty} \frac{(-1)^j \; \varphi_j(\rho)(1)}{j!} \qquad (3)$$

Define two formal series on $\mathfrak{gl}_n \oplus \mathfrak{gl}$:

$$(\mathrm{td}\,\mathcal{T})(X, Y) = \det\left[X(1 - e^{-X})^{-1}\right]; \quad (\mathrm{ch}\,\mathcal{E})(X, Y) = \mathrm{tr}\, e^{-Y}.$$

Theorem 4.1.2. $\mathrm{ch}(\rho) = \mathrm{ch}\,\mathcal{E} \cdot \mathrm{td}\,\mathcal{T}$.

4.2. Proof of Theorem 4.1.2.

Consider the composition

$$S^{n+q}(\mathfrak{gl}_n \oplus \mathfrak{gl})_{(\mathfrak{gl}_n \oplus \mathfrak{gl})} \xrightarrow{\sim} H_{2(n+q)}(W_*(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl})) \longrightarrow$$

$\longrightarrow H_{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; S^q D_n)$.

It may be describe in such a way. Let $v \in S^{n+q}(\mathfrak{gl}_n \oplus \mathfrak{gl})_{\mathfrak{gl}_n \oplus \mathfrak{gl}}$ and v' be its image in $W_{0,2(n+q)}$ under the inclusion $\mathfrak{gl}_n \oplus \mathfrak{gl} \xhookrightarrow{\rho} D_n$. If d is the differential in W_* then according to 3.1.1 there exists a chain $w \in W_*$ such that all the components of $v' + dw$ in $W_{i,*}$ are zero for $i < 2n$. Thus, the component of $v' + dw$ in $W_{2n,*}$ is a cycle in $C_*(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; S^k D_n)$. The corresponding homology class is the image of the above composition on v.

Consider the bigraded vector space

$$\widetilde{W}_* = W_*[[z_1, \dots, z_n; t_1, t_2, \dots]]$$

of formal power series with coefficients in W_*. Let $\widetilde{d} = (d \otimes 1)_{\mathbb{C}[[z_1,\dots,z_n;\ t_1,\ t_2,\dots]]}$. We obtain the bicomplex of $\mathbb{C}[[z_1,\dots,z_n;\ t_1,\ t_2,\dots]]$ - modules. Put

$$v^{(N)}(z, t) = \exp\Big(-\sum_{i=1}^{n} z_i x_i \partial_{x_i}\Big)\ \mathrm{diag}(e^{-t_1},\dots,e^{-t_N}, 0, 0,\dots) \qquad (4)$$

There exists such $w(z, t) \in \widetilde{W}_*$ that all components of $v^{(N)}(z, t) + \widetilde{d}w(z, t)$ in $\widetilde{W}_{i,*}$ are zero for $i < 2n$. Consider the cocycle $\sum \mu_j^* \omega_{\tau_n}$ and prolong it to $\widetilde{W}_{2n,*}$ by $\mathbb{C}[[z_1,\dots,z_n;\ t_1,\dots]]$ -linearity. It suffices to show that the value of this cocycle on $v_{2n}(z, t)$ is equal to $\prod z_i/(1 - e^{-z_i}) \cdot \sum e^{-t_k}$ where $v_{2n}^{(N)}(z, t)$ is the component of $v^{(N)}(z, t) + \widetilde{d}w(z, t)$ in $\widetilde{W}_{2n,*}$.

Note that we need only the case $N = 1$, $t_1 = 0$. Indeed,

$$v^{(N)}(z, t) = v^{(N)}(z, 0)\cdot\mathrm{diag}(e^{-t_1}, e^{-t_2},\dots, e^{-t_N}); \qquad (4)$$

as it will be shown below, $w(z, 0)$ may be chosen from the subcomplex $\widetilde{W}_*$ for the subalgebra $\mathrm{Diff}_n \cdot 1$. Since all the elements of this subalgebra commute with $\mathfrak{gl}$, one may choose

$$w(z, t) = w(z, 0)\cdot\mathrm{diag}(e^{-t_1}, e^{-t_2}, \dots, e^{-t_N}),$$

and thus

$$v_{2n}^{(N)}(z, t) = v_{2n}^{(N)}(z, 0)\cdot\mathrm{diag}(e^{-t_1}, \dots, e^{-t_N});$$

$$v_{2n}^{(N)}(z, 0) = \sum f_{i_1\dots i_{2n+1}}(z_1,\dots,z_n)(X_{i_1}1\wedge\dots\wedge X_{i_{2n}}1)\otimes X_{i_{2n+1}}1 \tag{5}$$

where $X_{i_1},\dots, X_{i_{2n}} \in \mathrm{Diff}_n$, $X_{i_{2n+1}} \in S^{**}D_n$; it is easy to see that such a cycle is homologous in $C_*(D_n, gl_n \oplus gl; S^{**}D_n)$ to a cycle

$$v'_{2n}(z, t) = \sum_k\sum e^{-t_k} f_{i_1\dots i_{2n}i_{2n+1}}(X_{i_1}E_{11}\wedge\dots\wedge X_{i_{2n}}E_{11})\otimes X_{i_{2n+1}}E_{11};$$

thus

$$\left\langle \sum \mu_j^* \omega_{\tau_n}, v_{2n}^{(N)}(z, t)\right\rangle = \left\langle \sum \mu_j^* \omega_{\tau_n}, v_{2n}^{(1)}(z, 0)\right\rangle \sum_{k=1}^{N} e^{-t_k} \tag{6}$$

So, we must consider the case $N = 1$, $t_1 = 0$. At first, suppose that $n = 1$. Denote for simplicity $E_{11}\cdot X$ by X. Put $L_j = x^{j+1}\partial$, $j \geqslant -1$. Put $v(z) = v^{(1)}(z, 0)$, $v_2(z) = v_2^{(1)}(z, 0)$. We have $v(z) = e^{-zL_0}$. Represent e^{-zL_0} as an image under the differential $\widetilde{W}_{1,*} \to \widetilde{W}_{0,*}$. One has

$$e^{-zL_0} - 1 = \sum_{m=1} \left[L_m, L_{-1}^m \Phi_m(L_0)\right] \tag{7}$$

where

$$\Phi_{m+1}(L_0) = \frac{(-1)^{m+1}}{(m+1)(m+2)!}\left(\frac{\partial}{\partial L_0}\right)^m \left(\frac{e^{-zL_0} - 1}{L_0}\right);$$

thus, $e^{zL_0} = \widetilde{d}_1 w$ where

$$w = \sum L_m \otimes L_{-1}^m \Phi_m(L_0)$$

(recall that $\widetilde{d} = \widetilde{d}_1 + \widetilde{d}_2$, $\widetilde{d}_1 : \widetilde{W}_{i,*} \to \widetilde{W}_{i-1,*}$, $\widetilde{d}_2 : \widetilde{W}_{i,m} \to \widetilde{W}_{i+1,m-2}$).
Applying $\widetilde{d}_2$ to w one obtains

$$v_2(z) = \sum_{m=0} \frac{(-1)^{m+1}}{(m+2)!} (L_{m+1} \wedge L_{-1}) \otimes \left(\frac{\partial}{\partial L_0}\right)^m \frac{e^{-zL_0} - 1}{L_0} L_{-1}^m +$$

$$+ (\partial \wedge x) \otimes 1 \qquad (8)$$

For any differential operator $X = \sum h_\ell \partial^\ell$ put $X_\ell = h_\ell$. Lemma 3.3.1 implies that

$$\langle \mu_j^* \omega_{\tau_n}, (L_{m+1} \wedge L_{-1}) \otimes Y = -m!\, \mu_j (Y)_m (0)$$

So we have

$$\frac{\partial}{\partial z} \left\langle \sum_{j \geq 0} \mu_j^* \omega_{\tau_n}, v_2(z) \right\rangle = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)} \mu\left(\left(\frac{\partial}{\partial L_0}\right)^m e^{-zL_0} L_{-1}^m\right)_m (0)$$

where $\mu = \sum_{j \geq 0} \mu_j$.

We shall use the following Lemma.

Lemma 4.2.1. Let φ be a function and ψ satisfy the relation $\psi^{(n)} = \varphi$. Then

$$\mu(\varphi(L_0) \cdot L_{-1}^m)_m (0) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \psi(k).$$

Proof may be obtained by straightforward verification.

It follows from the Lemma that

$$\frac{\partial}{\partial z} \left\langle \sum_{j \geq 0} \mu_j^* \omega_{\tau_n}, v_2(z) \right\rangle = - \sum_{m=0}^{\infty} \frac{(1 - e^{-z})^m}{(m+1)(m+2)} ;$$

denote the right hand side by $U(\ )$. We have

$$\frac{d}{dz} ((1 - e^{-z})^2 U(z)) = z e^{-z};$$

on the other hand,

$$\frac{d}{dz} \left((1 - e^{-z})^2 \cdot \left(\frac{z}{1 - e^{-z}}\right)'\right) = z e^{-z};$$

comparing the values in zero we obtain

$$(1 - e^{-z})^2 U(z) = (1 - e^{-z})^2 \left(\frac{z}{1 - e^{-z}}\right)' ;$$

once more comparing the values in zero we have

$$\Big\langle \sum_{j\geq 0} \mu_j^* \omega_{\tau_n},\ v_2(z)\Big\rangle = z/(1 - e^{-z}). \tag{9}$$

This ends the proof for the case n = 1.

Now we pass the the general case. If $X = \sum h_\ell \partial^\ell \in \mathrm{Diff}_1$ we put $X^{(i)} = \sum h_\ell(x_i)\partial_{x_i}^\ell \in \mathrm{Diff}_n$; if $w_i = (X_i \wedge Y_i) \otimes Z_i \in W_{2,*}(D_1, \mathfrak{gl}_1 \oplus \mathfrak{gl})$ then

$$w_1 \otimes \dots \otimes w_n = X_1^{(1)} \wedge Y_1^{(1)} \wedge \dots \wedge \underbrace{X_n^{(n)}}_{\wedge Y_n^{(n)} \wedge} Z_n^{(n)} \quad Z_1^{(1)} \dots Z_n^{(n)} \in$$

$$\in W_{2n,*}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}). \tag{10}$$

We obtain a map $W_{2,*}(D_1, \mathfrak{gl}_1 \oplus \mathfrak{gl})^{\otimes n} \to W_{2n,*}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl})$. Analogously, changing z by z_i at the i-th place, we define a map

$$W_{2,*}(D_1, \mathfrak{gl}_1 \oplus \mathfrak{gl})^{\otimes n} \to \widetilde{W}_{2n,*}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}).$$

It is easy to see that we may choose

$$v_{2n}(z, 0) = v_2(z, 0)^{\otimes n} \tag{11}$$

Furthermore, as we have discussed in 3.3, the cocycle $\tau_n = \tau^{\otimes n}$ is a basis element of $HH^{2n}(\mathrm{Diff}_n)$. It follows from the implicit formula for exterior multiplication in HH^* (cf. [CE]) that

$$\tau_n(X_1^{(1)}, Y_1^{(1)}, \dots, X_n^{(n)}, Y_n^{(n)}, Z_1^{(1)} \dots Z_n^{(n)}) = \tau(X_1, Y_1, Z_1) \dots \tau(X_n, Y_n, Z_n)$$

$$\tau_n(X_1^{(i_1)}, Y_1^{(i_1)}, \dots, X_n^{(i_n)}, Y_n^{(i_n)}, Z) = 0, \quad (i_1 \dots i_n) \neq (1 \dots n)$$

these formulas together with the formula for ω_{τ_n} from 3.1 imply that

$$\Big\langle \sum_j \mu_j^* \omega_{\tau_n},\ v_{2n}^{(1)}(z, 0) = \sum_j \mu_j^* \omega_{\tau_n},\ v_2(z)^{\otimes n}\Big\rangle =$$

$$= \prod_{i=1}^n \Big\langle \sum_j \mu_j^* \omega_\tau,\ v_2(z_j)\Big\rangle = \prod_{i=1}^n \frac{z_i}{1 - e^{-z_i}}. \tag{12}$$

This ends the proof of Theorem 4.2.1.

§5. Absolute local Riemann-Roch theorem

5.1. First, we recall the well known construction of characteristic classes (cf., for example, [F]).

Let L be a Lie algebra and $\mathfrak{g}$ be a subalgebra reductive in L. Let $\theta : L \to \mathfrak{g}$ be a projection operator which is $\mathfrak{g}$-equivariant. Consider the curvature form

$$\Theta(X, Y) = \theta([X, Y]) - [\theta(X), \theta(Y)].$$

This is a $\mathfrak{g}$-equivariant skew symmetric $\mathfrak{g}$-values 2-form on $L/\mathfrak{g}$ satisfying the equation $d\Theta + [\Theta, \theta] = 0$. For $P \in S^k(\mathfrak{g}^*)^{\mathfrak{g}}$ let

$$c_p = P(\underbrace{\Theta, \ldots, \Theta}_{(k \text{ times})}); \qquad c_p \in C^{2k}(L, \mathfrak{g}; \mathbb{C}).$$

It may be shown that this construction provides a characteristic homomorphism which does not depend on θ:

$$S^*(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{c} H^{2*}(L, \mathfrak{g}; \mathbb{C}) \tag{1}$$

Now let $L = D_n$, $\mathfrak{g} = \mathfrak{gl}_n \oplus \mathfrak{gl}$ (see above). Let $\mathrm{ch}\,\mathcal{E}$, $\mathrm{td}\,\mathcal{T}$ be the elements of $S^{**}(\mathfrak{g}^*)^{\mathfrak{g}}$ defined in 4.1. Let $(\mathrm{ch}\,\mathcal{E}\cdot\mathrm{td}\,\mathcal{T})_n$ be the component of $c(\mathrm{ch}\,\mathcal{E}\cdot\mathrm{td}\,\mathcal{T})$ in H^{2n}. On the other hand, consider the module inclusion $i : \mathbb{C} \to D_n$ and the dual map $D_n^* \to \mathbb{C}$; consider also the basis element $\omega_{\tau_n} \in H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; D_n^*)$ defined in 3.3.

<u>Theorem 5.1.1</u>. $\frac{(-1)^n}{n!}\, i^*\omega_{\tau_n} = (\mathrm{ch}\,\mathcal{E}\cdot\mathrm{td}\,\mathcal{T})_n$ in $H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; \mathbb{C})$.

<u>Proof</u>. Consider the maps

$$S^n(\mathfrak{gl}_n^* \oplus \mathfrak{gl}^*)^{\mathfrak{gl}_n \oplus \mathfrak{gl}} \xleftarrow{\sim} H^{2n}(W^*(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl})) \to H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; \mathbb{C}) \tag{2}$$

The map on the right is the edge homomorphism to $E_1^{2n,0}$. It is an isomorphism because $E_1^{ij} = 0$ for $i < 2n$. So, one has an isomorphism:

$$S^n(\mathfrak{gl}_n^* \oplus \mathfrak{gl}^*)^{\mathfrak{gl}_n \oplus \mathfrak{gl}} \xrightarrow[\sim]{c'} H^{2n}(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; \mathbb{C}) \qquad (3)$$

We shall show that this isomorphism coincides with (1). Choose a projection operator Θ as follows. Put for $m \in \mathfrak{gl}(\mathbb{C})$, $X = f\,\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$:

for $\sum \alpha_i = 0$ $\quad \Theta(Xm) = f(0)\cdot m;$

for $\sum \alpha_i = 1$ and $\deg f \neq 1, \quad \Theta(Xm) = 0;$

for $\sum \alpha_i = 1$ and $\deg f = 1, \quad \Theta(Xm) = \operatorname{tr} m\cdot X\cdot 1;$

for $\sum \alpha_i > 1, \quad \Theta(Xm) = 0.$

Let $w = X_1 \wedge \dots \wedge X_{2n} \otimes Y$ be a chain of $C_*(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; S^* D_n)$. Put

$$\tilde{c}^*(w) = \sum_{\substack{\sigma \in S_{2n} \\ \sigma(2k-1) < \sigma(2k)}}{}' \operatorname{sgn}\sigma \cdot \Theta(X_{\sigma 1}, X_{\sigma 2}) \dots \Theta(X_{\sigma(2n-1)}, X_{\sigma(2n)})\cdot Y$$

The map c^* dual to c is the restriction of $\tilde{c}^*$ to $C_*(D_n, \mathfrak{gl}_n \oplus \mathfrak{gl}; S^0)$. To show that $c'^* = c$ it suffices to show that

$$c^* v_{2n}^{(N)}(\mathfrak{z}, t) = v^{(N)}(\mathfrak{z}, t) \qquad (4)$$

(in notation of 4.2). But it follows from the formulas (8)-(12) of 4.2 together with the equalities:

$$\Theta(\partial_{x_i}, x_i) = 1; \quad \Theta(L_1^{(i)}, L_{-1}^{(i)}) = -2L_0^{(i)};$$

$$\Theta(L_m^{(i)}, L_{-1}^{(i)}) = 0, \quad m \neq 1;$$

$$\Theta(X^{(i)}, Y^{(j)}) = 0, \quad i \neq j.$$

Now Theorem 5.1.1 follows from 4.1.2. ∎

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