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RATIONAL HOMOTOPY TYPE OF NILPOTENT AND COMPLETE SPACES*

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Introduction.

Sullivan [4,10] and Bousfield-Gugenheim [2] proved the equivalence of the rational homotopy category of nilpotent spaces with rational homology of finite type and the homotopy category of differential graded rational algebras with minimal models of finite type. Earlier, Quillen [9] proved the equivalence of the rational homotopy category of simply connected spaces with two categories (among others): the homotopy category of simply connected differential graded rational coalgebras and the homotopy category of connected differential graded Lie algebras. Neisendorfer [7] combined the approaches from [9,4,10] to nilpotent spaces with rational homotopy of finite type. Unsöld [11, 12] used topological graded algebras to prove results similar to those given in [9,4,10], without the finite type restriction, but for the simply connected case only.

The object of this paper is to give a generalisation of the works of Neisendorfer [7] and Unsöld [11,12] on rational complete spaces in the sense of [3]. In §3 the main result is established. Namely the rational homotopy category of complete spaces is equivalent to the homotopy category of: topological differential graded rational algebras, differential graded rational coalgebras and nilpotent completion graded rational Lie algebras.

In §1 we recall some basic definitions associated with linearly topological vector spaces. In §2 we generalise the Bousfield-Gugenheim result [2] and prove that the category of connected complete differential rational algebras is a closed model category in the sense of Quillen [8]. Moreover, in §3 we prove that Q-complete spaces are closed with respect to iteration. By this result we partially answer the following question of Baues [1]: Are R-Postnikov spaces of order 2 closed with respect to iteration?

The proofs of our results and applications to the rational homotopy theory will be developed in forthcoming papers. It would be inte-

*) "This paper is in final form and no version of it will be submitted for publication elsewhere".

resting to know if similar results also occur in the tame homotopy theory given in [5].

1. Topological vector spaces

Let k be a discrete field of characteristic zero. A topological k-vector space V is called linearly topological [6] if it is Hausdorff and there is a fundamental system of neighbourhoods $\mathcal{V}$ consisting of nuclear (i.e. open-closed) vector subspaces of V. This system $\mathcal{V}$ is called a linear topology on V. A morphism of linearly topological spaces $V \to W$ is a continuous homomorphism of vector spaces V and W. We denote by $Vect_k$ the resulting category of linearly topological spaces.

Let V' be a vector subspace of a linearly topological space V and let V/V' denote the quotient topological vector space. If V' is a closed subspace of V then V/V' is also a linearly topological space. In particular, if V' is nuclear then V/V' is discrete. If V' and V'' are two nuclear subspaces of V such that $V' \subseteq V''$ then we obtain the canonical morphism $V/V' \to V/V''$. For a linear topology $\mathcal{V}$ on V we put $\hat{V} = \lim_{\mathcal{V}} V/V'$. Then $\hat{V}$ is a linearly topological space with respect to the topology determined by spaces V/V', for all V'. This space $\hat{V}$ is called the completion of V. The system of canonical morphisms $V \to V/V'$ for all V' determines an injective morphism $V \to \hat{V}$. If this morphism is an isomorphism then V is called a complete linearly topological space. A linearly topological space V is called linearly compact if any filterbasis $\mathcal{F}$ of V consisting of affine subspaces has a clusterpoint, i.e. $\bigcap_{F \in \mathcal{F}} \bar{F} = \emptyset$.

According to Köthe [6] we have the following results.

Proposition 1.1 (i) A closed subspace of a linearly compact space is linearly compact.

(ii) The image of a linearly compact space is linearly compact.

(iii) Products and inverse limits of linearly compact spaces are linearly compact.

(iv) Each linearly compact space is complete.

(v) A linearly compact space is discrete if and only if it is finitely dimensional.

Let now V and W be linearly topological spaces. Let C be a linearly compact subspace of V and W' be a nuclear subspace of W. We denote by $N(C,W')$ the subspace of all morphisms $f: V \to W$ such that $f(C) \subseteq W'$. The system $\{N(C,W')\}$ determines a linear topology on $Vect_k(V,W)$ for all linearly compact subspaces C of V and all nuclear subspaces W' of W. This topology is called the linear compact-open topology. Therefore, we have defined an internal hom functor $VECT_k(-,-)$ on the category $Vect_k$. In particular, we obtain a linear topology on the topological dual vector space $V^* = Vect_k(V,k)$. The functor $VECT_k(V,-)$ has a left adjoint, a tensor product $- \otimes V$. If W is a linear topological space, then $V \otimes W$ is topologised by all subspaces $[V', W'] = V \otimes W' + V' \otimes W$, for nuclear subspaces V' of V and W' of W. We denote by $V \hat{\otimes} W$ the completion $(V \otimes W)^\wedge$.

Proposition 1.2 ([6]) (i) A linearly topological space V is linearly compact if and only if V^* is discrete.

(ii) If V is linearly compact or discrete then $V \simeq V^{**}$.

(iii) If V and W are linearly compact then $VECT_k(V,W) \simeq VECT_k(W^*, V^*)$ and $(V \hat{\otimes} W)^* \simeq V^* \otimes W^*$.

(iv) If V is linearly compact and W is discrete then $V^* \otimes W \simeq VECT_k(V,W)$.

Thus it follows that the functor * establishes an equivalence of the categories of linearly compact and discrete spaces.

2. Topological algebras

Let $A = \bigoplus_{n \geq 0} A^n$ be a differential graded k-algebra. We assume all our algebras to be augmented and commutative in the graded sense. We call A complete [12] if:

(i) A^n is a complete space for all $n \geq 0$;

(ii) multiplication $\mu : A^n \times A^m \longrightarrow A^{n+m}$ is uniformly continuous;

(iii) differential $d: A^n \longrightarrow A^{n+1}$ is continuous.

A is said to be linearly compact if A^n is linearly compact for all $n \geq 0$. We denote by ZA, BA and HA the group of cocycles, coboundaries and cohomologies of A respectively. A morphism of complete algebras $A \to B$ is a continuous homomorphism of differential graded k-algebras. We denote by Dga_k or by Dga_k^0 the resulting category of complete or connected complete k-algebras. A morphism $A \to B$ in Dga_k is said to be a weak equivalence if the induced maps $HA \longrightarrow HB$ are isomorphisms. A morphism $E \to B$ in Dga_k is called a fibration if maps $E^* \longrightarrow B^*$ are surjective. Trivial fibrations are morphisms which are both fibrations and weak equivalences. Following Quillen [8] we define a cofibration in Dga_k to be a morphism which has the left lifting property with respect to trivial fibrations. Trivial cofibrations are morphisms which are both cofibrations and weak equivalences. An algebra A in

Dga_k is called <u>cofibrant</u> if the unique morphism $k \to A$ is a cofibration.

Now we describe a particular kind of cofibrations in Dga_k. Let $V = \bigoplus_{n \geq 1} V^n$ be a connected graded linearly topological space and let ΛV denote the free graded algebra on V.

<u>Proposition 2.1</u> ([12]) There is a linear topology on ΛV such that the usual universal map property holds with respect to continuous maps.

We denote by FV the completion $(\Lambda V)^{\wedge}$. Then FV is a complete graded algebra in the sense of our definition. If V is linearly compact in each degree then FV is linearly compact and $(FV)^*$ is isomorphic to $\Lambda(V^*)$. If V is discrete then ΛV is also discrete, and so $\Lambda V = FV$.

Let A be an algebra in Dga_k and let d denote its differential. For a linearly compact space V (not graded) we denote by (V,n) the graded space given by V in degree n and 0 otherwise. Let also $t : V \to Z^{n+1}A$ be a morphism of linearly topological spaces. Then by [12] there is a differential d_t on $A \hat{\otimes} F(V,n)$ such that $d_t|A = d$ and $d_t|V = t$. The algebra $A \hat{\otimes} F(V,n)$ with the differential d_t is called an <u>elementary extension</u> of A of degree n. Let $A \to A \hat{\otimes} F(V,n)$ be the canonical inclusion.

We call a cofibration $j : A \to D$ <u>nilpotent</u> when there is a tower

$$A = D^0 \to D^1 \to \dots \to D^{k-1} \to D^k \to \dots \to D$$

of elementary extensions $D^{k-1} \to D^k$ $(k \geq 1)$ with an isomorphism $h : \operatorname{colim}_k D^k \to D$ under A.

We call j a <u>nilpotent completion</u> when there is a tower

$$A = D^0 \to D^1 \to \dots \to D^{k-1} \to D^k \to \dots \to D$$

with each D^k is a colimit of elementary extensions of degree k starting from D^{k-1} $(k \geq 1)$ and an isomorphism $h : \operatorname{colim}_k D^k \to D$ under A.

It may be noted that each nilpotent cofibration is a nilpotent completion. Nilpotent or nilpotent completion cofibrations are called <u>minimal nilpotent</u> or <u>minimal nilpotent completion</u> k-algebras respectively.

Following [11] we say that an algebra $A = \bigoplus_{n \geq 0} A^n$ in Dga_k satisfies the condition (M) if:

(M1) the following sequences split topologically

$$0 \to Z^*A \to A^* \to B^{*+1}A \to 0$$

and

$$0 \to B^*A \to Z^*A \to H^*A \to 0$$

(M2) HA are linearly compact spaces.

We denote by $Dga_k(M)$ the full subcategory og Dga_k consisting of algebras satisfying this condition (M).

The following statements generalise the Bousfield-Gugenheim results from [2].

Proposition 2.2 Let A be a linearly compact k-algebra and $f : A \longrightarrow B$ be a morphism in $Dga_k(M)$ such that $H^0 f$ is an isomorphism and $H^1 f$ is injective. Then there is a unique nilpotent completion $M_f : A \longrightarrow D$ and a weak equivalence $h : D \longrightarrow B$ under A. In particular, each algebra A in $Dga_k(M)$ is weakly equivalent to a unique minimal nilpotent completion M_A.

The cofibration M_f or the algebra M_A is called the minimal nilpotent completion model of f or of A respectively. We denote by Min_k the category consisting of minimal nilpotent completion k-algebras and by $Min_k(1)$ its full subcategory generated by minimal nilpotent k-algebras. It may be noted that each object of Min_k is linearly compact.

Moreover we have

Theorem 2.3 The category Dga_k^0 together with the distinguished classes of weak equivalences, fibrations and cofibrations defined above satisfies the Quillen closed model axioms (cf. [8]).

We denote by $HoDga_k^0$ the resulting homotopy category and by $HoDga_k^0(M)$ its full subcategory generated by objects of Dga_k^0. Let now Dgc_k^0 and $Dgla_k$ denote the category of connected differential graded k-coalgebras and Lie k-algebras respectively. Let $ncDgla_k$ be the full subcategory of $Dgla_k$ generated by Lie k-algebras L such that HL is a nilpotent completion (according to [7]). But by [7] (Proposition 7.2) their homotopy categories in the sense of Quillen [8] are equivalent. Thus we obtain the following generalisation of this approach.

Proposition 2.4 The following three homotopy categories $HoDga_k^0(M)$, $HoDgc_k^0$ and $Ho\ ncDgla_k$ are equivalent.

3. Sullivan-de Rham theorem

We regard Q, the field of rationals, as a discrete space. Let S^0 be the category of 0-reduced simplicial sets and let HoS^0 be its homotopy category according to [8]. Following [1], p.110 we distinguish in S^0 a class of fibrations.

Let π be a Q-module and let $K(\pi,n)$ $(n \geq 1)$ be the Eilenberg-MacLane simplicial set. We call a principal fibration $E \longrightarrow B$ a Q-Postnikov simplicial set over B (of <u>order 0</u>). We call a fibration $E \longrightarrow B$ a Q-<u>Postnikov</u> simplicial set over B (of <u>order n</u>, $n \geq 1$) when there is a tower

$$E \to \dots \to E^k \to E^{k-1} \to \dots \to E^0 = B$$

of Q-Postnikov simplicial sets $E^k \longrightarrow E^{k-1}$ (of order n-1, $k \geq 1$) and an isomorphism $E \longrightarrow \lim_k E^k$ over B.

We denote by $S^0_Q(n)$ ($n \geq 0$) the category of rational simplicial sets X such that there is a Q-Postnikov simplicial set $E \to *$ over the final object $*$ in S^0 (of order n) and a simplicial map $X \longrightarrow E$ that induces isomorphism $H(E,\pi) \longrightarrow H(X,\pi)$ for all Q-modules π. Objects of $S^0_Q(n)$ we call Q-<u>Postnikov</u> simplicial sets (of <u>order n</u>). In particular, for n = 1 and n = 2 we obtain Q-<u>nilpotent</u> or Q-<u>complete</u> simplicial sets defined in [3].

In [1], p.111 Baues asks the following question: Are R-Postnikov spaces (of order 2) closed with respect to iteration? For R = Q, we partially answer this question. Namely we conclude from a generalisation of the Sullivan-de Rham theorem [2] that Q-complete simplicial sets are closed with respect to iteration. Let X be a simplicial set and let $A^*(X)$ be the algebra of Q-polynomial forms on X (cf. [2]). Following [12] we topologise $A^*(X)$ as follows: for any siplicial map $\bar{x} : \Delta^m \longrightarrow X$ the subspace $Ker(A^n(\bar{x}) : A^n(X) \longrightarrow A^n(\Delta^m))$ is nuclear in $A^n(X)$ for all $m,n \geq 0$, where Δ^m denote the standard m-simplex. Then $A^*(X)$ is a complete Q-algebra satisfying the condition (M). Let M_X denote its minimal nilpotent completion model. Then the rule $X \mapsto M_X$ determines a functor $M : HoS^0 \longrightarrow HoMin^0_Q$. For an algebra A in Dga^0_Q let GA be the simplicial set given by $(GA)_n = Dga^0_Q(A, A^*(\Delta^n))$ for all $n \geq 0$. It is easy to show that A and G are adjoint functors

$$A : S^0 \rightleftarrows Dga^0_Q : G.$$

Using Theorem 2.3 it can be shown that the following generalisation of the Sullivan- de Rham theorem [2] holds.

<u>Theorem 3.1</u> The functors M and G induce a pair of adjoint functors

$$\bar{M} : HoS^0_Q(1) \rightleftarrows HoMin^0_Q(1) : \bar{G}$$

and for $n \geq 2$ $\qquad \bar{M} : HoS^0_Q(n) \rightleftarrows HoMin^0_Q : \bar{G}$

which are inverse the one to the another.

In particular, we obtain

<u>Corollary 3.2</u> Q-complete simplicial sets are closed with respect to iteration (i.e. $HoS^0_Q(n) \simeq HoS^0_Q(2)$ for $n \geq 2$).

The next proposition generalises the Neisendorfer result from [7] (Proposition 7.3).

Proposition 3.3 The following four homotopy categories are equivalent $HoS^O_Q(n)$ $(n\geq 2)$, $HoDga^O_Q(M)$, $HoDgc^O_Q$ and $HoncDgla_Q$.

The question arises which does not appear to have been answered yet. Do similar results also occur in the tame homotopy theory introduced in [5]?

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